Exact Real Trinomial Solutions to the Inner and Outer Hele–Shaw Problems

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Exact Real Trinomial Solutions to the Inner and Outer Hele–Shaw Problems

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Abstract. We find an explicit representation of the evolution of $t \mapsto \gamma_t = \{z(\zeta,t),\ \zeta \in \mathbb{C},\ |\zeta| = 1\}$ of the contour $\gamma_t = \partial\omega_t$ of fluid spots $\omega_t = \{z(\zeta,t),\ |\zeta| < 1\}$ for $t > 0$ or $t < 0$ in the Hele–Shaw problem with a sink ($t > 0$) or a source ($t < 0$) localized at point $z(0,t)$ described by trinomials

$$z(\zeta,t) = a_1(t)\zeta + a_N(t)\zeta^N + a_M(t)\zeta^M, \text{where} \quad M = 2N-1, \quad \text{and integer} \quad N \geq 2,$$

for the classical formulation of the problem when ω_t is within γ_t (inner Hele–Shaw problem), or by

$$z(\zeta,t) = a_{-1}(t)\zeta^{-1} + a_N(t)\zeta^N + a_M(t)\zeta^M, \text{where} \quad M = 2N+1, \quad \text{and integer} \quad N \geq 1,$$

for the outer Hele–Shaw problem when ω_t is outside of γ_t. We obtained a sufficient condition for univalence of real trinomials, improving a result found by Ruscheweyh and Wirths (Ann Pol Math. 28:341–355, 1973). A sufficient condition is also found for functions used in the outer problem.

Mathematics Subject Classification. Primary 76B07; Secondary 30C50.

Keywords. Hele–Shaw cell, exact real trinomial solution, domain of univalence.

1. Introduction

1.1. Hele–Shaw Flows and Stokes–Leibenson Model

Considering the slow flow of a fluid between two parallel flat plates separated by a infinitesimally small gap, Hele–Shaw wanted to visualize stream lines of the flow around an object (particularly around the hull of a ship). Working on this experiment, he accidentally discovered that the transition between laminar and turbulent flow did not take place for any velocity. The same year as the publication by Hele–Shaw of his work in Nature [8], a modelisation in two dimensions of the problem was proposed by Stokes [20] and Lamb [13].

We consider a configuration in which two immiscible fluids fill the plane; one of them is the *active* phase and has the geometry of a simply connected domain. Navier–Stokes equations are reduced to a Laplace's equation $\Delta p = 0$, whose unknown p represents the pressure within the active phase. On the boundary between the phases, time evolution of the contour is realized by a Darcy's law. For further theoretical treatments, Leibenson [14] simplifies the problem, neglecting surface tension on the boundary (fixing $p = 0$). This also means that the viscosity of the considering (active) fluid is much greater that viscosity of the surrounding fluid. Without source or sink of fluid, the solution $p = 0$ in the fluid domain trivially verifies all the equations. In most applications (injection moulding or oil extraction), fluid is injected or extracted through a small hole, represented by a Dirac delta function (or an equivalent near the sink or source point) and belonging to the active phase.

We can sum up the content of the previous paragraph into a system of three equations (see 1.1). The associated Fig. 1 gives an example of an initial domain with its boundary in the inner and outer

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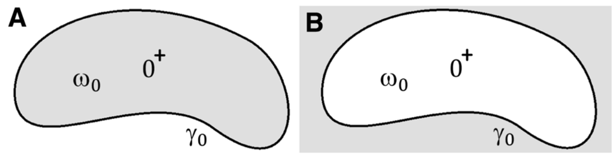

FIG. 1. An example of initial fluid domain in inner (**a**) and outer (**b**) Hele–Shaw problem

Hele–Shaw problem. The domain ω_0 corresponds to an initial fluid domain in plane $z\,(z = x + \imath y)$, which will be deformed according to the following laws: at time $t \geq 0$, we obtain a domain ω_t with a boundary γ_t, which moves at velocity $\dot{\mathbf{s}}$ at point $\mathbf{s}$, such that

$$\begin{array}{lll} \textit{Stokes condition:} & \Delta p = 0\,, & \text{in } \omega_t\,, \\ \textit{Leibenson dynamical condition:} & p = 0\,, & \text{on } \gamma_t\,, \\ \textit{Kinetic condition (Darcy's law):} & \dot{\mathbf{s}} \cdot \nu = -\partial_\nu p\,, & \text{on } \gamma_t. \end{array} \tag{1.1}$$

p is the pressure in the fluid domain ω_t, ν the outward-pointing unit normal vector and $\partial_\nu p$ the normal outward derivative of p.

Fluid is extracted uniformly at a fixed rate (that is for $t > 0$ in sink-case) at point z_0 ($z_0 = 0$ for the inner problem, $z_0 = +\infty$ for the outer one), which corresponds to the additional condition for points z near z_0

$$p \sim C\, \ln(|z|), \quad \text{as } |z| \to z_0.$$

Constant C equals to $+1$ in the inner problem and -1 in the outer one.

The mathematical treatment of these equations should be considered with specific mathematical tools.

1.2. Conformal Transformation

The first theoretical achievement in Hele–Shaw problem was proposed by Galin [6] and Polubarinova–Kochina [16]. Using a conformal map between the unit disk U and the fluid domain ω_t, they succeed in extracting an equation governing the shape of the fluid boundary in Stokes–Leibenson approximation. Discovered in 1945, this equation remains the main tool for studying the Hele–Shaw flows, in particular because of the amount of exact solutions it admits.

A conformal transformation locally conserves the angles and the shapes of infinitesimally small figures. In particular, a conformal transformation can be realized by a complex function f from the unit disk to the current domain $f(U)$ if this function is holomorphic and its derivative is everywhere non-zero on U. This leads to the Riemann mapping theorem: a non-empty open simply connected domain in $\mathbb{C}$ admits a bijective conformal map to the open unit disk in $\mathbb{C}$. The configuration of this theorem is represented on the Fig. 2.

We will use the fact that if a function f is univalent in U then the derivative of f is never zero in U.

In the condition of the theorem, we have, for a family of univalent holomorphic functions indexed by time t, $f(\cdot, t) : \{|\zeta| < 1\} \to \omega_t$ in both inner and outer problems, the Polubarinova–Galin equation, which takes the form

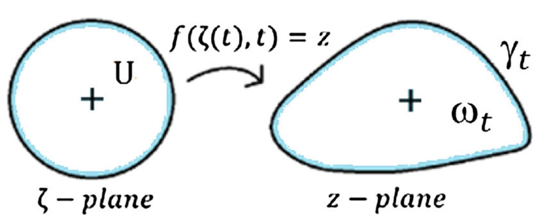

FIG. 2. Riemann mapping theorem

$$\Re\left(\frac{\partial f}{\partial t}(\zeta,t)\zeta\overline{\frac{\partial f}{\partial \zeta}(\zeta,t)}\right) = -1, \quad \text{with} \quad |\zeta| = 1, \tag{1.2}$$

where we chose to describe the sink-case when time increases. Local existence of a solution takes place when f is analytic or only smooth in the source-case (i.e. $t < 0$) ([7,15,21]). The unicity is provided by fixing the image of two points, one in the domain, the second on the boundary.

1.3. The Search of Exact Solutions

In the publications by Galin [6] and Polubarinova–Kochina [16] are described the system of equations obtained by a polynomial solution of degree N ($N \in \mathbb{N}$, $N > 1$) for the inner problem. In the case of a binomial solution of type $a_1(t)\zeta + a_N(t)\zeta^N$, $|\zeta| = 1$, with a_1, $a_N \in \mathbb{R}$ they explicit an exact solution. We have two following first integrals

$$\frac{1}{2}a_1(t)^2 + \frac{1}{2}Na_N(t)^2 = -t + C, \quad a_1(t)^N a_N(t) = D,$$

where constants C and D are determined by the initial conditions.

After this pioneer work, a lot of exact solutions were found: rational and logarithmic by Kufarev and Vinogradov [22], hypergeometric by Howison and King [10], self similar by Ben Amar [1], or travelling-wave by Saffman and Taylor [19]... However nothing else was done on polynomial solutions, except that trajectories for a trinomial solution were obtained by Huntingford in 1995 [11] for the inner problem, only in particular case $N = 2$. His work was quickly forgotten, so quickly that I discovered it when I was just finishing writing this paper. The idea of reducing the quantity of unknowns to only two remains the same as in his article, however I did not use resolution with Schwartz function but direct calculations with Polubarinova–Galin equation, which allows to write a simple proof for all trinomials. Furthermore, it is possible to understand how fast points move on trajectories, which give exact solutions. The same proof can be done with functions describing the outer configuration.

Definition 1.1. Trinomials used in the inner Hele–Shaw problem are called trinomials of type 1, whereas trinomials in the outer problem of type 2.

Particular attention have to be given to the understanding of the domain of univalence for trinomials of type 1 and 2. This difficulty can explain the lack of interest in polynomial solutions. This question is the content of the Sect. 2.

2. Domain of Univalence

2.1. Presentation of the Trinomial Case

A family of complex functions indexed by unknowns $A_1, \ldots, A_P$ ($P \in \mathbb{N}^*$) in the unit disk defines a domain of univalence in the P-dimensional space $(A_1, \ldots, A_P)$. Particularly, for the trinomials, this domain is a region of the plane, generated by the two coefficients of a univalent normalized trinomials.

Univalent functions are those that can physically represent the border of a fluid. Consequently the graphical representation of $\gamma = \partial\omega$ should be a simple closed curve. Two configurations have to be avoided: the presence of singularities on γ (created when a cusp occurs) and the presence of double points (created when the curve comes into contact with itself).

To study the domain of univalence of trinomial solutions, we introduce normalized trinomial functions defined on the closed unit disk into domain ω by

$$f_{in}(\zeta) = \zeta + \frac{Y}{N}\zeta^N + \frac{X}{2N-1}\zeta^{2N-1}, \quad \text{with} \quad N \in \mathbb{N}\backslash\{0, 1\},$$

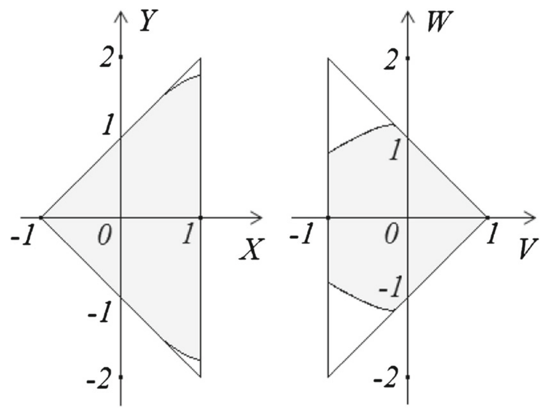

FIG. 3. Domain of univalence in the inner problem with $N = 3$ (*left*) and outer problem with $N = 2$ (*right*) in *grey*

and

$$f_{out}(\zeta) = \frac{1}{\zeta} + \frac{W}{N}\zeta^N + \frac{V}{2N+1}\zeta^{2N+1}, \quad \text{with} \quad N \in \mathbb{N}^*,$$

where the two couples of real numbers, (X, Y) and (V, W), define two planes in which we search to precise the domain of univalence for families of functions f_{in} and f_{out}.

The first configuration with cusps corresponds to an impossibility to solve the Eq. (4.1) due to an annulation of the derivative of f_{in} or f_{out} (see further Eq. 4.2). It leads to necessary conditions of univalence: points (X,Y) (and (V,W)) should belong to the interior of a triangle, represented on the Fig. 3. When a point crosses the border of the triangle, a loop appears (or disappears) going through a singularity in the graphical representation of γ.

Trinomials with double points are localized in a neighborhood of points $(X, Y) = (1, \pm 2)$ and $(V, W) = (-1, \pm 2)$ belonging to the triangles. For all considered trinomials, a trivial symmetry can be seen, and we restricted our study to positive half-planes $Y \geq 0$ and $W \geq 0$. The curves $Y = c_N(X)$ and $W = d_N(V)$ separate the univalent domain from the domain with double points (cf Fig. 3). These domains can not (yet) be explicitly described (except for c_2 and c_3 see [4] and [17]), however some informations exist, due to a result by Ruscheweyh and Wirths [18].

Theorem 2.1 (see [18] for c_N). *The domain of univalence of the function $f_{in}(\cdot)$ and $f_{out}(\cdot)$ are respectively*

- $U(f_{in}) = \{-c_N(X) < Y < c_N(X), \quad |X| < 1\} \subset T_{in}$ *where*

$$T_{in} = \{-(X+1) < Y < X+1, \quad |X| < 1\},$$

and the function $c_N : [-1, 1] \to [0, 2]$ is continuous and increases monotonically. Besides,

$$c_N(X) = X + 1 \quad \mathit{for} \quad X \in \left[-1, \frac{N+1}{3N-1}\right],$$

and

$$c_N\left(\frac{N+1}{3N-1}\right) = \frac{4N}{3N-1}, \quad c_N(1) = \frac{2N^2}{2N-1}\sin\frac{\pi}{2N};$$

- $U(f_{out}) = \{-d_N(V) < W < d_N(V), \quad |V| < 1\} \subset T_{out}$ *for the outer problem where*

$$T_{out} = \{V - 1 < W < -V + 1, \quad |V| < 1\},$$

and the function $d_N : [-1, 1] \to [0, 2]$ is continuous. Besides,

$$d_N(V) = -V + 1 \quad \mathit{for} \quad V \in \left[-\frac{N-1}{3N+1}, 1\right],$$

and

$$d_N\left(-\frac{N-1}{3N+1}\right)=\frac{4N}{3N+1},\quad d_N(-1)=\frac{2N^2}{2N+1}\sin\frac{\pi}{2N}.$$

Moreover, d_N is increasing on $[-1,V_{max}]$, decreasing on $\left[V_{max},-\dfrac{N-1}{3N+1}\right]$ with

$$V_{max}=\cos\frac{\pi}{2N+1}-\frac{N\sin\dfrac{\pi}{2N+1}}{\tan\dfrac{N\pi}{2N+1}}\quad \text{and}\quad d_N(V_{max})=N\frac{\sin\dfrac{\pi}{2N+1}}{\sin\dfrac{N\pi}{2N+1}}.$$

Functions d_1 and d_2 can be explicitly described (see the proof).

As far as we know, this is the best theorem for c_N on trinomial polynomials f_{in}. Concerning functions d_N, we did not find any results in literature, so that we have to prove the results.

2.2. Characterisation of the d_N Functions

Most of the works about univalence of polynomials are based on a result by Dieudonné [5] and a rule by Cohn [3]. We give these two results and then established the proof of the second part of the Theorem 2.1, taking inspiration from [18].

Dieudonné's Criterion: Let $f(\zeta)=\zeta+a_2\zeta^2+\cdots+a_N\zeta^N$ be a normalized complex polynomial. Function f is univalent in the unit disk U if and only if the associated equation

$$1+\sum_{k=2}^{N}a_k\frac{\sin(k\theta)}{\sin(\theta)}z^{k-1}=0,$$

has no roots in the unit disk ($|z|<1$) for any $\theta\in[0,\frac{\pi}{2}]$.

Cohn's Rule: If $\sum_{k=0}^{N}a_k\zeta^N$ is a polynomial of degree N and $|a_0|>|a_N|$ then it has the same number of roots in $|\zeta|<1$ as the polynomial

$$\overline{a_0}f(\zeta)-a_N\zeta^N\overline{f(1\backslash\overline{\zeta})}.$$

For the study of d_N we need to introduce a new kind of polynomials slightly different of the classical ones. We define the "polynomial" f^* by

$$f^*(\zeta)=\frac{1}{\zeta}+a_1\zeta+\cdots+a_N\zeta^N.$$

A modified Dieudonné's criterion can be set using this form of polynomial. This variant of this criterion was already found by Brannan [2].

Lemma 2.2 (Modified Dieudonné's Criterion)*: Function f^* is univalent in $0<|\zeta|<1$ if and only if the associated equation*

$$-1+\sum_{k=1}^{N}a_k\frac{\sin(k\theta)}{\sin(\theta)}z^{k+1}=0,$$

has no roots in the unit disk ($|z|<1$) for any $\theta\in\left[0,\frac{\pi}{2}\right]$.

With polynomials f_{out} we have

$$-1+\frac{W}{N}\frac{\sin(N\theta)}{\sin(\theta)}z^{N+1}+\frac{V}{2N+1}\frac{\sin((2N+1)\theta)}{\sin(\theta)}z^{2(N+1)}=0,\quad N\in\mathbb{N}^*.$$

If $\left|\frac{V}{2N+1}\frac{\sin((2N+1)\theta)}{\sin(\theta)}\right| \geq 1$, then the product of the roots is less than 1 and at least one root belongs to the unit disk. Thus, because of the univalence of f_{out} we can apply the Cohn's rule; knowing that the coefficients are real, we obtain the condition

$$|A| \leq \min_{\theta \in \left[0, \frac{\pi}{2}\right]} \frac{1 - B\dfrac{\sin(2N+1)\theta}{\sin\theta}}{\left|\dfrac{\sin N\theta}{\sin\theta}\right|} = \min_{\theta \in \left[0, \frac{\pi}{2}\right]} G_N(\theta, B),$$

where $A = \dfrac{W}{N}$ and $B = \dfrac{V}{2N+1}$.

2.2.1. Study of Small Degrees

Case $N = 1$.

We have $G_1(\theta, B) = 1 + B - 4B\cos^2\theta$, so that the boundary of the univalent domain can be explicitly described by relation

$$|W| \leq 1 - \frac{V}{3} - 2\,\mathrm{sign}(V)\frac{V}{3}, \quad V \in [-1, 1].$$

Case $N = 2$.

In case $N = 2$, we obtain $G_2(c, B) = \frac{1}{2c} - B\left(2c(4c^3 - 3) + \frac{1}{2c}\right)$, where $c = \cos\theta \in]0, 1]$. The study of this function shows us that the minimum is reached at point $\tilde{c} = \sqrt{\frac{1}{8} + \frac{3}{8}\sqrt{\frac{7B-4}{3B}}}$ when $B \in \left[-\frac{1}{5}, -\frac{1}{35}\right]$ and point $\tilde{c} = 1$ when $B \in [-\dfrac{1}{35}, \dfrac{1}{5}]$.
Thus,

$$|W| \leq \begin{cases} G_2\left(\tilde{c}, \dfrac{V}{5}\right) & \text{if} \quad -1 \leq V < -1/7, \\ 1 - V & \text{if} \quad -1/7 \leq V \leq 1. \end{cases}$$

2.2.2. Study for $N \geq 3$

Lemma 2.3. *For* $B \in \left[-\frac{1}{2N+1}, \frac{1}{2N+1}\right]$*, the function* $G_N(\theta, B)$ *attains its minimum for* θ *in the interval* $\left[0, \frac{\pi}{2N}\right]$.

Proof. We notice that

$$G_N\left(\frac{\pi}{2N}, B\right) \leq \left(1 + \frac{1}{2N+1}\right)\sin\frac{\pi}{2N} \leq \frac{N+1}{N}\frac{\pi}{2N+1},$$

and we set that

$$G_N(\theta, B) > \frac{N+1}{N}\frac{\pi}{2N+1}, \quad \text{for}\ \theta \in \left[\frac{\pi}{2N}, \frac{\pi}{2}\right].$$

Indeed, for $\theta \in \left[\dfrac{\pi}{2N}, \dfrac{\pi}{2}\right]$ we have for $N \geq 3$ the following inequalities

$$-\frac{2N+1}{3} \leq \frac{\sin(2N+1)\theta}{\sin\theta} \leq \frac{2N+1}{3},$$

and

$$\left|\frac{\sin N\theta}{\sin\theta}\right| \leq \left(\sin\frac{\pi}{2N}\right)^{-1}.$$

Thus,

$$G(\theta,B) \geq \left(1 - \frac{1}{2N+1}\frac{2N+1}{3}\right)\sin\frac{\pi}{2N} = \frac{2}{3}\sin\frac{\pi}{2N} > \frac{N+1}{N}\frac{\pi}{2N+1}.$$

The minimum should be search in the interval $\theta \in \left[0, \frac{\pi}{2N}\right]$ in which we have

$$G_N(\theta,B) = \frac{\sin\theta}{\sin N\theta} - B\frac{\sin(2N+1)\theta}{\sin N\theta}.$$

□

2.2.3. Study of G_N. We differentiate the function G_N by θ,

$$\begin{aligned} h(\theta) &= \frac{\partial G_N}{\partial\theta}(\theta,B)\,\sin^2 N\theta \\ &= (\cos\theta - (2N+1)B\cos(2N+1)\theta)\sin N\theta - (\sin\theta - B\sin(2N+1)\theta)N\cos N\theta \end{aligned}$$

and

$$\frac{\partial h}{\partial\theta} = ((N^2-1)\sin\theta + (3N+1)(N+1)B\sin(2N+1)\theta)\sin N\theta,$$

furthermore,

$$h(0) = 0 \quad \text{and} \quad h\left(\frac{\pi}{2N}\right) = (1+(2N+1)B)\cos\frac{\pi}{2N} \geq 0.$$

The unique non-zero root $\tilde{\theta}$ of $\dfrac{\partial h}{\partial\theta}$ verifies the equation

$$-\frac{N-1}{(3N+1)B} = \frac{\sin(2N+1)\tilde{\theta}}{\sin\tilde{\theta}}. \tag{2.1}$$

Unicity came from the fact that function $\frac{\sin(2N+1)\theta}{\sin\theta}$ is strictly decreasing on $\left[0, \frac{\pi}{2N}\right]$ and it is clear that on $]0,\tilde{\theta}[$ the function h is negative and on $\left]\tilde{\theta}, \frac{\pi}{2N}\right[$ positive. The expression (2.1) is majorated by $2N+1$ on the considered interval and we can derive a condition of existence of a unique root for $\frac{\partial h}{\partial\theta}$ in $\left]0, \frac{\pi}{2N}\right[$, that is to say for the existence of a unique minimum on $\left]0, \frac{\pi}{2N}\right[$ for the function G_N

$$-1 < (2N+1)B = V < -\frac{N-1}{3N+1}.$$

Consequently, for $(2N+1)B = V \in \left[-\frac{N-1}{3N+1}, 1\right]$, we obtain

$$\min_{\theta\in\left[0,\frac{\pi}{2}\right]} G_N(\theta,B) = \frac{1-(2N+1)B}{N}.$$

We define function $\tilde{\theta}(B)$ such that

$$G_N(\tilde{\theta}(B),B) = \min_{\theta\in\left[0,\frac{\pi}{2}\right]} G_N(\theta,B),$$

and we search in which point B we have $\frac{\partial G_N}{\partial B}(\tilde{\theta}(B),B) = 0$. This point is obtained when the expression $\frac{\sin(2N+1)\theta}{\sin N\theta}$ is equal to zero for $\theta \in \left]0, \dfrac{\pi}{2N}\right[$. The only one solution is $\theta = \frac{\pi}{2N+1}$. Then, the equation $\frac{\partial G_N}{\partial\theta}\left(\frac{\pi}{2N+1}, B\right) = 0$ can be solved and we have

$$(2N+1)B = V = \cos\frac{\pi}{2N+1} - \frac{N\sin\dfrac{\pi}{2N+1}}{\tan\dfrac{N\pi}{2N+1}}.$$

At this point, the curve d_N reaches its maximum and takes the value

$$W_{max} = NA_{max} = N\frac{\sin\dfrac{\pi}{2N+1}}{\sin\dfrac{N\pi}{2N+1}}.$$

Notice that $W_{max} \to \frac{\pi}{2}$ when $N \to +\infty$.

For $V = -1$ we have $h(0) = h\left(\frac{\pi}{2N}\right) = 0$ and the minimum is reached in one of this two points. A simple calculation shows us that the right choice is $\theta = \frac{\pi}{2N}$ and we get the result described in the Theorem 2.1. The proof is then completed.

3. Results

We consider real trinomial solutions to the Hele–Shaw problem with initial conditions given at $t = 0$ (belonging to the domain of univalence) of the following form, for the inner formulation,

$$f_1(\zeta, t) = a_1(t)\zeta + a_N(t)\zeta^N + a_M(t)\zeta^M, \quad \text{where} \quad M = 2N-1, \quad \text{and} \quad N \geq 2,$$

and, for the outer formulation,

$$f_{-1}(\zeta, t) = a_{-1}(t)\zeta^{-1} + a_N(t)\zeta^N + a_M(t)\zeta^M, \quad \text{where} \quad M = 2N+1, \quad \text{and} \quad N \geq 1.$$

This kind of trinomials is stable by the Polubarinova–Galin equation, that's why no other harmonic occurs during time evolution (for $t > 0$ or $t < 0$).

The main idea for solving the problem with trinomial functions of type 1 and 2 is to introduce new unknowns

$$X = M\frac{a_M}{a_1}, \qquad Y = N\frac{a_N}{a_1}, \qquad \text{for the inner problem,}$$

and

$$V = M\frac{a_M}{a_{-1}}, \qquad W = N\frac{a_N}{a_{-1}}, \qquad \text{for the outer problem.}$$

This normalization is justified by the fact that time evolution of the boundary does not depend on initial area. The resulting system of equations can be partially solved, leading to the following results.

Theorem 3.1. *For trinomial initial contours in inner problem, the solution moves on a curve, whose equation is*

$$Y = C_{in}\frac{|X|^{\dfrac{N+1}{2N}}}{1+X},$$

with $C_{in} = Y_0(1+X_0)|X_0|^{-\frac{N+1}{2N}}$.

The notation with a small zero corresponds to initial values in the domain of univalence $\left(e.g. X_0 = \frac{(2N-1)a_{2N-1}(0)}{a_1(0)}\right)$.

In the sink-case, the point (X,Y) reaches in finite time the border of the domain of univalence. In the source-case, the point approaches the origin $\mathbf{O}$ *and can possibly exists for all time (see Theorem 3.2).*

Time evolution (and consequently area evolution $|\Omega_t|$) is described by relation

$$|\Omega_0| - t = |\Omega_t| = \frac{1}{2}\left(\frac{\alpha}{X}\right)^{\frac{1}{N}} \left(1 + \frac{Y^2}{N} + \frac{X^2}{2N-1}\right),$$

where

$$\alpha = (2N-1)a_{2N-1}(0)a_1^{2N-1}(0).$$

For trinomial initial contours in outer problem, we have the same kind of result. Curve is given by relation

$$W = C_{out}\frac{|V|^{\frac{N-1}{2N}}}{1-V},$$

with $C_{out} = W_0(1-V_0)|V_0|^{\frac{1-N}{2N}}$. Area evolution $|\Omega_t|$ is described by

$$|\Omega_0| + t = |\Omega_t| = \frac{1}{2}\left(\frac{V}{\beta}\right)^{\frac{1}{N}} \left(1 - \frac{W^2}{N} - \frac{V^2}{2N+1}\right),$$

where $\beta = \frac{(2N+1)a_{2N+1}(0)}{a_1^{2N+1}(0)}$.

We would like to understand for which initial conditions the contour could exist for all time in source-case. This is the content of the following theorems.

Theorem 3.2. *For the inner problem, if initial condition occurs above the curve*

$$\tilde{Y} = \frac{|X|^{\frac{N+1}{2N}}}{1+X}\frac{(1+X_0)^2}{|X_0|^{\frac{N+1}{2N}}}$$

with $X_0 = \dfrac{N+1}{3N-1}$, the contour meets a singularity in the source-case.

The same result occurs for $\tilde{W} = \frac{|V|^{\frac{N-1}{2N}}}{1-V}\frac{(1-V_0)^2}{|V_0|^{\frac{N-1}{2N}}}$ with $V_0 = -\frac{N-1}{3N+1}$ in the outer problem.

We further demonstrate that all points located below curve $\tilde{Y}$ (or $\tilde{W}$) move on curves, without encountering any points of c_N (respectively d_N). This leads to the following theorem.

Theorem 3.3. *For $X \in \left[\frac{N+1}{3N-1}, 1\right]$, function c_N satisfies the inequality $\tilde{Y}(X) \leq c_N(X) \leq \hat{Y}(X)$ where*

$$\hat{Y}(X) = \frac{4N^2}{2N-1}\sin\left(\frac{\pi}{2N}\right)\frac{|X|^{\frac{N+1}{2N}}}{1+X}.$$

Furthermore, we have $\lim_{N\to+\infty}(\hat{Y}(1) - \tilde{Y}(1)) = \dfrac{\pi}{2} - \dfrac{8}{9}\sqrt{3} < 0.032$.

For $V \in \left[-1, -\frac{N-1}{3N+1}\right]$, function d_N satisfies the inequality $\tilde{W}(V) \leq d_N(V) \leq \hat{W}(V)$ where

$$\hat{W}(V) = \frac{4N^2}{2N+1}\sin\left(\frac{\pi}{2N}\right)\frac{|V|^{\frac{N-1}{2N}}}{1-V}.$$

Furthermore, we have $\lim_{N\to+\infty}(\hat{W}(1) - \tilde{W}(1)) = \dfrac{\pi}{2} - \dfrac{8}{9}\sqrt{3} < 0.032$.

4. Proof of the Theorems

In order to not overload the text, we chose to prove theorems only in the inner case. Results for outer flows could be easily obtained with small modifications of the equations.

4.1. Proof of Theorem 3.1

Substituting f in the Eq. (1.2) by a trinomial function of type 1 ($f(\zeta,t) = a_1(t)\zeta + a_N(t)\zeta^N + a_M(t)\zeta^M$ with $M = 2N-1$ $(N>1)$), we explicit the system of equations

$$\begin{pmatrix} a_1 & Na_N & Ma_M \\ Na_N & a_1 + Ma_M & Na_N \\ Ma_M & 0 & a_1 \end{pmatrix} \begin{pmatrix} \dot{a_1} \\ \dot{a_N} \\ \dot{a_M} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

This system can be transformated into a Cauchy problem in which arises the determinant

$$\begin{pmatrix} \dot{a_1} \\ \dot{a_N} \\ \dot{a_M} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} -a_1(a_1 + Ma_M) \\ Na_N(a_1 - Ma_M) \\ Ma_M(a_1 + Ma_M) \end{pmatrix}, \tag{4.2}$$

where $D = (a_1 - Ma_M)(a_1 + Ma_M + Na_N)(a_1 + Ma_M - Na_N)$.

We now introduce the unknowns X and Y, which leads to the equations

$$\begin{cases} \dot{X} = \dfrac{1}{a_1^2 E} 2NX(1+X), \\ \dot{Y} = \dfrac{1}{a_1^2 E} Y(X(1-N) + N + 1), \end{cases}$$

with $E = (1-X)(1+X+Y)(1+X-Y)$. Noticed that

$$\frac{\dot{Y}}{\dot{X}} = \frac{Y(X(1-N)+N+1)}{2NX(1+X)},$$

we can separate the expression into two simple ordinary differential equations (losing the linear description of the time)

$$\begin{cases} \dot{Y} = Y, \\ \dot{X} = \dfrac{2NX(1+X)}{X(1-N)+N+1}. \end{cases}$$

The second equation can be reduced to $\frac{N+1}{2N}\frac{\dot{X}}{X} - \frac{\dot{X}}{1+X} = 1$, which gives the result after integration.

Time evolution of the area is obtained by integration of the first and third equations in (4.1).

4.2. Proof of Theorem 3.2

Among the trajectories which have common points with equation $Y = 1 + X$, we would like to know if one of them can be tangential to it and, at this point, remains locally in the triangle.

To this end, we solved $\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}X} = 1$, with $\tilde{Y}$ a function Y obtained in the previous theorem such that $\tilde{Y}(X_0) = 1 + X_0$. We have

$$\frac{N+1}{2N}\frac{1+X_0}{X_0} - 1 = 1,$$

so that

$$X_0 = \frac{N+1}{3N-1}.$$

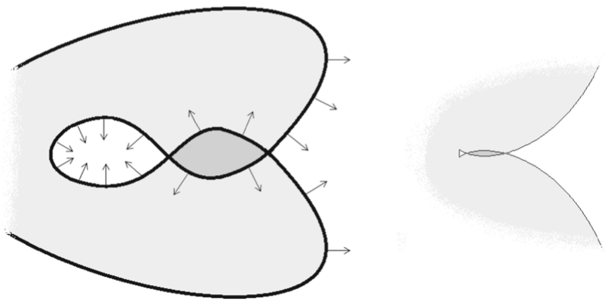

FIG. 4. *Left*: schematization of the dynamic of a contour with a loop by Polubarinova–Galin equation, *right*: example of a contour with double points with a trinomial of type 1 given by equation $\zeta + \frac{1.88}{2}\zeta^2 + \frac{0.95}{3}\zeta^3$

Looking at the derivative of function $\tilde{Y}$, we conclude that the curve $\tilde{Y}$ remains locally around point $X = X_0$ in the interior of the triangle. Notice that $\tilde{Y}$ is a strictly increasing function in $[0, 1]$.

For the solution $\tilde{Y}$ a blow-up occurs at finite time (at point X_0), but the solution continues to exist. The same kind of behavior was first highlighted by Howison in [9].

4.3. Proof of Theorem 3.3

To prove that for all initial points below curve $\tilde{Y}$ the contour remains univalent during time evolution (in source-case) without possible occurring of a double point, we look at the Polubarinova–Galin equation and its properties. The left side of this equation represents the quantity $\overrightarrow{V} \cdot \overrightarrow{n}$ where $\overrightarrow{V}$ is the velocity of a point of the boundary and $\overrightarrow{n}$ the outer normal vector and we know that $\overrightarrow{V} \cdot \overrightarrow{n} > 0$ in source-case.

We prove this result by contradiction, supposing that an initial contour below curve $\tilde{Y}$ represents a contour with two double points. The Polubarinova–Galin equation remains usable for contours with a loop (we do not cross a cusp, thus the behaviour of the contour remains the same by continuity), although the dynamic did not represent in this case the dynamic of a 2D fluid but takes place on a riemannian geometry. Double points in the fluid can be considered as belonging to two different layers of a thin screw in order to conserve Riemann mapping transformation. On Fig. 4 we see that the presence of a loop leads to a cusp because of property $\overrightarrow{V} \cdot \overrightarrow{n} > 0$.

The removing of these two double points can be realized only by going through two cusps: when time increases, boundary tends to a circle (see the trajectories of the solutions), then the apparition of these two cusps is inevitable. We have set that, if a double point occurs, then the next singularity to appear is a cusp. This is impossible for an initial contour below $\tilde{Y}$.

The second inequality with function $\hat{Y}$ comes from the same argument, knowing that $c_N(1) = \frac{2N^2}{2N-1} \sin\left(\frac{\pi}{2N}\right)$.

5. Visualisation of the Results

On the following Fig. 5, we represent trajectories of solutions in the inner problem for various initial conditions. The grey zone corresponds to the domain above c_N within the triangle, that is $T_{in} \backslash U(f_{in})$. Another zone exists but is too small to be represented, this is a “forbidden” area in $U(f_{in})$ near c_N $\left(X > \frac{N+1}{3N+1}\right)$, in which the contour (if initial point represents an univalent trinomial) meets a singularity in the source-case. We remind that the symmetry of solutions in order to axe (Ox) justifies the restriction to the domain $Y \geq 0$.

In the article by Kuznetsova [12] was obtained a sufficient condition for all time existence of a polynomial solution. In the particular case of a trinomial, this condition leads to the relation

$$Y_0^2 + \frac{N}{2N-1} X_0^2 < \frac{N}{(N-1)(2N+1)}.$$

The Theorem 3.3 is a necessary and sufficient condition for the case of trinomial solutions of type 1 and 2.

Remark 5.1. The area of the Kuznetsova's domain tends to 0 when N tends to infinity, while our domain fills more than 99,95 % of the domain of univalence when N tends to infinity.

For the outer problem, the same kind of figure for trajectories can be represented in the plane (U, W) (see Fig. 6).

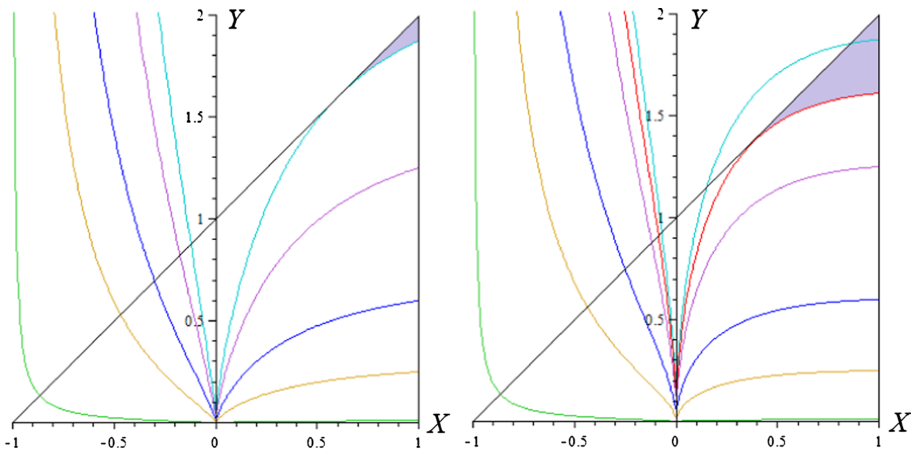

Fig. 5. Trajectories of the inner problem for various initial conditions. On the *left* $N = 2$, on the *right* $N = 10$

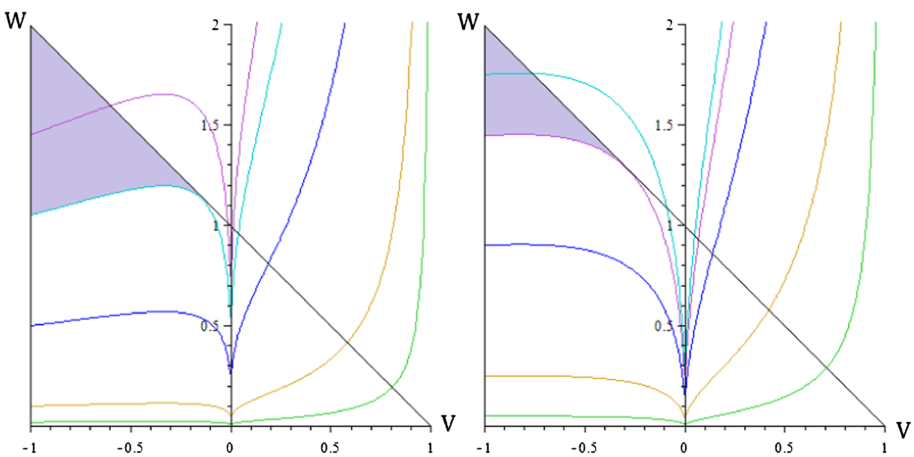

Fig. 6. Trajectories of the outer problem for various initial conditions. On the *left* $N = 2$, on the *right* $N = 10$

To conclude this paper, we chose to represent contours at different time of its evolution. We give an example for the inner problem and another one for the outer problem on Figs. 7 and 8.

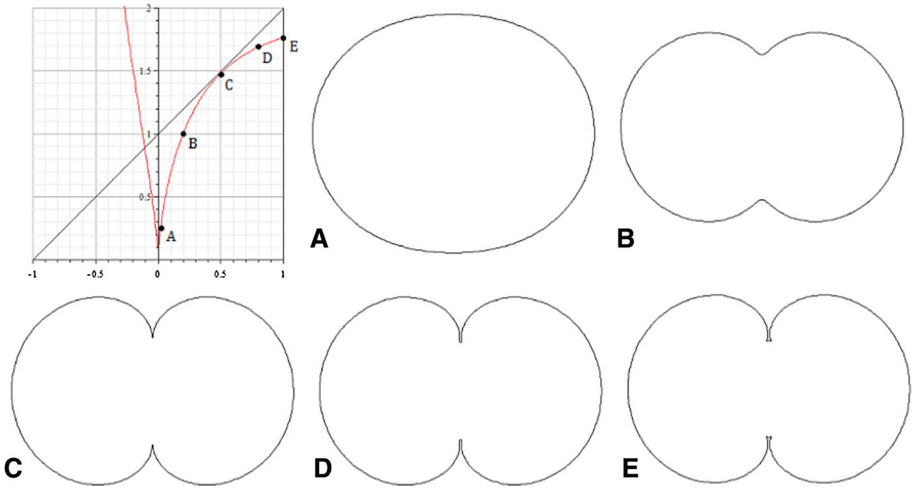

FIG. 7. Time evolution in the sink-case of a trinomial of type 1 with $N = 2$ from point A to E

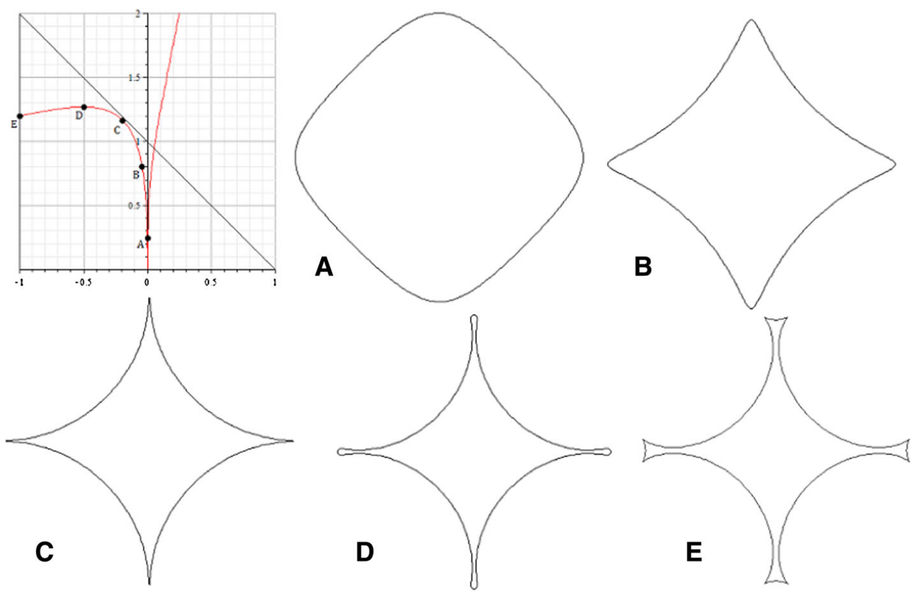

FIG. 8. Time evolution in the sink-case of a trinomial of type 2 with $N = 4$ from point A to E

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