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Optimal Control by Multipoles in the Hele-Shaw Problem

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Abstract. The two-dimensional Hele-Shaw problem for a fluid spot with free boundary can be solved using the Polubarinova–Galin equation. The main condition of its applicability is the smoothness of the spot boundary. In the sink-case, this problem is not well-posed and the boundary loses smoothness within finite time—the only exception being the disk centred on the sink. An extensive literature deals with the study of the Hele-Shaw problem with non-smooth boundary or with surface tension, but the problem remains open. In our work, we propose to study this flow from a control point of view, by introducing an analogue of multipoles (term taken from the theory of electromagnetic fields). This allows us to control the shape of the spot and to avoid non-smoothness phenomenon on its border. For any polynomial contours, we demonstrate how all the fluid can be extracted, while the border remains smooth until the very end. We find, in particular, sufficient conditions for controllability and a link between Richardson's moments and Polubarinova–Galin equation.

Mathematics Subject Classification. Primary 49K15; Secondary 76D27.

Keywords. Hele-Shaw flow, multipoles, controllability, optimal control.

1. Introduction

1.1. Limit in Laplace's Equation

This first part of this article is dedicated to a theoretical result, allowing us to simplify further limit computations in the Laplace's equation. In this article, we make all the limit calculations in two different ways: by using Theorem 1.1 or by the classical way. Consequently, Theorem 1.1 strongly simplifies calculations but can be skiped in a first reading.

Let us consider the following problem

$$\begin{cases} \Delta p = S, \\ p|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

with Ω a simply connected and bounded domain in $\mathbb{C}$ with smooth boundary $\partial\Omega$, p a unknown function (physically, a pressure) and S a generalized real-valued function with $\operatorname{supp} S \Subset \Omega$, i.e. $\in \mathcal{E}'(\Omega)$. The function S is a *source-term*, which contains all information about sources/sinks and multipoles.

The first equation in (1.1) has a solution $p \in \mathcal{S}'(\mathbb{C})$ for any $S \in \mathcal{E}'(\mathbb{C})$ (see [6], p. 69, theorem 3.2.1). Also any generalized solution of equation $\Delta u = 0$ in any domain is a smooth analytic function (see [6], p. 101, corollary 4.1.2, and p. 114, corollary 4.4.1). So any solution p of the first equation is smooth in a neighbourhood of $\partial\Omega$. Consequently $p|_{\partial\Omega}$ is a smooth function, and the second condition in (1.1) is well posed.

Let $E(z) = \dfrac{1}{2\pi} \ln |z|$ denote a fundamental solution for the Laplace operator $\Delta E = \delta_0$.

Theorem 1.1. *There exists a unique solution of problem* (1.1) *and it can be found as* $p_S = E * S - \varphi_{E*S}$, *with* $*$ *being the convolution between functions, and where* φ_{E*S} *denotes a smooth analytic solution of the problem*

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$$\begin{cases} \Delta\varphi = 0, \\ \varphi|_{\partial\Omega} = (E * S)|_{\partial\Omega}. \end{cases}$$

Moreover the map $\Phi\colon S \mapsto p_S$ *is linear and continuous on* $\mathcal{E}'(\Omega_1)$ *for any* $\Omega_1 \Subset \Omega$.

Proof. We know that $E*S \in \mathcal{S}'(\mathbb{C})$ is a solution of the first equation in (1.1). We set that $p_S = E*S-\varphi_{E*S}$ is the unique solution of problem (1.1). Let p_1 and p_2 be two solutions of (1.1). Then $p_2 - p_1$ is a solution of (1.1) with $S = 0$ and $p_2 - p_1$ is a smooth analytic function (see [5], p. 101, corollary 4.1.2), therefore $p_2 - p_1 = 0$ by maximum principle, so that uniqueness is proved.

The map Φ being clearly linear, we only have to set the continuity of Φ. Let us consider the map $L\colon \mathcal{E}'(\Omega_1) \to C^\infty(\partial\Omega)$, $L\colon S \mapsto (E * S)|_{\partial\Omega}$. We prove that L is continuous. We denote by $a > 0$ the distance between $\partial\Omega$ and Ω_1 and choose an $\varepsilon \in (0, a)$ and define $A = \{a - \varepsilon \le |z| \le \operatorname{diam}\Omega + \varepsilon\}$. Then we can decompose E in the sum

$$E = E_1 + E_2 \quad \text{where } E_1(z) = E(z), \quad \forall z \in A, \quad \text{and} \quad \operatorname{supp} E_2 \cap A = \emptyset.$$

We may assume that $E_1 \in C_0^\infty(\mathbb{C})$ because $0 \notin A$.

We claim that $(E_1 * S)|_{\partial\Omega} = (E * S)|_{\partial\Omega}$ for all $S \in \mathcal{E}'(\Omega_1)$. Indeed $E * S = E_1 * S + E_2 * S$. On the one hand $\operatorname{supp}(E_2 * S) \subset \operatorname{supp} E_2 + \operatorname{supp} S \subset (\mathbb{C}\backslash A) + \Omega_1$, and on the other hand $\Omega_1 - \big(\partial\Omega + \{|z| < \varepsilon\}\big) \subset A$. So $\operatorname{supp}(E_2 * S) \cap \big(\partial\Omega + \{|z| < \varepsilon\}\big) = \emptyset$.

The convolution

$$L_1\colon \mathcal{E}'(\Omega_1) \to C_0^\infty(\mathbb{C}), \quad L_1\colon S \mapsto E_1 * S,$$

is continuous. The restriction

$$L_2\colon C_0^\infty(\mathbb{C}) \to C^\infty(\partial\Omega), \quad L_2\colon \psi \mapsto \psi|_{\partial\Omega},$$

is continuous. So $L = L_2 \circ L_1$ is also continuous.

Consequently, the map $M\colon \mathcal{E}'(\Omega_1) \to C^\infty(\Omega) \cap C(\overline{\Omega})$, $M\colon S \mapsto \varphi_{E*S}$, is continuous by maximum principle. The theorem is then proved by $\Phi(S) = E * S - M(S)$. □

Remark 1.2. The continuity of Φ gives us the possibility to make easy passage to limits in problem (1.1)

$$\Phi\left(\lim_{n\to\infty} S_n\right) = \lim_{n\to\infty} \Phi(S_n).$$

The sufficient condition is that $\operatorname{dist}(\operatorname{supp} S_n, \partial\Omega) > \varepsilon$ where $\varepsilon > 0$ does not depend on n.

Example. If $0 \in \Omega$ and $S_n = (\delta_{\frac{1}{n}} - \delta_0)/n$ then the limit solution p of (1.1) as $n \to \infty$ is the solution of (1.1) with $S = \delta_0'$, i.e. $p = E * \delta_0' - \varphi_{E*\delta_0'}$. The first term is easy to compute: $E * \delta_0' = (E * \delta_0)' = E' = \frac{1}{2\pi}\Re\left(\frac{1}{z}\right)$. The second term may be explicitly found as a solution of the Dirichlet problem. For example let $\Omega = \{|z| < 1\}$. Then on $\{|z| = 1\}$ we have $\Re\left(\frac{1}{z}\right) = \Re(z)$ and $\varphi_{E*\delta_0'} = \frac{1}{2\pi}\Re(z)$. Therefore $p = \frac{1}{2\pi}\Re\left(\frac{1}{z} - z\right)$.

From now, the calculus of limits in the Laplace problem will be done indifferently on the solution or on the equation. The part 2.1 named *Dipole and multipole approximation* consists in an explanation about the building of multipoles, but strong justifications are already established by Theorem 1.1.

1.2. Generalized Polubarinova–Galin Equation

We consider a slow flow between two parallel flat plates and made up of two unmiscible fluids. The viscosity of the first fluid is much more higher than the viscosity of the second one. In such conditions, the Navier–Stokes equations can be reduced to simple 2D equations only living in the first phase [8]. This phase is geometrically a simply connected domain ω_0 at initial moment, and will be deformed in a domain ω_t in time evolution according to the following laws.

At time $t \ge 0$, we obtain a domain ω_t whose boundary is $\gamma_t = \partial\omega_t$. Each point $\mathbf{s}$ on ω_t moves at velocity $\dot{\mathbf{s}}$ along the direction of the outward-pointing unit normal vector $\nu(\mathbf{s})$, such that

$$kinetic\ condition\ (Darcy\text{'s}\ law\ [1])\colon \dot{\mathbf{s}} \cdot \nu = -\partial_\nu p, \quad \text{on } \gamma_t,$$

$\partial_\nu p$ being the normal outward derivative of p. Pressure p is determined by the inner state of the phase

$$\begin{aligned} &\textit{Stokes condition } [15]\colon -\Delta p = S, \quad \text{in } \omega_t,\\ &\textit{Leibenson dynamical condition } [10]\colon p = 0, \quad \text{on } \gamma_t. \end{aligned}$$

Function $S = S(z)$ is the *source term* in ω_t made of pointwise contributions. In this article, S will be a combination of Dirac delta functions and its derivatives localized in a neighbourhood of point $\mathbf{O}$.

Remark 1.3. It is important to notice that these equations may be written as long as the boundary γ_t remains smooth.

The above-described configuration is called *a Hele-Shaw problem in Stokes–Leibenson approximation*. An effective way to translate these equations into an elegant mathematical relation is to use an idea proposed by Polubarinova–Kochina and Galin in 1945 (see [3,12]). Using a conformal transformation, they succeeded in obtaining a unique equation governing the shapes of the fluid boundary. An important remark was made concerning a lost of regularity in the pointwise sink-case. Indeed, in any case (except the circle centered on the sink), solution ceases to exist before extracting all the fluid [4] (lost of regularity of the contour). The proof of this equation is based on the Riemann mapping theorem, which establises the existence of a holomorphic bijection f between the unit disk and a simply connect domain. Function f is unique modulo conformal transformations from the unit disk to itself.

As we would like to use different kind of source, we need a generalized version of the Polubarinova–Galin equation. We introduce the notation

$$\dot{f} = \frac{\partial f}{\partial t}(\zeta, t) \quad \text{and} \quad f' = \frac{\partial f}{\partial \zeta}(\zeta, t),$$

and we have.

Theorem 1.4. *Generalized Polubarinova–Galin equation.*

For an univalent function $f(\cdot, t)\colon \{|\zeta| < 1\} \to \omega_t$ from the unit disk to the current domain, Polubarinova–Galin equation takes the form[1]

$$\Re\big(\dot{f}\ \overline{\zeta\ f'}\big) = -\frac{\partial W}{\partial \zeta}\zeta, \quad |\zeta| = 1, \tag{1.2}$$

with $W = W(\zeta, t)$ solution of the Dirichlet problem in the unit disk $U = \{|\zeta| < 1\}$:

$$\begin{aligned} &\textit{Stokes condition}\colon -\Delta_\zeta W = |f'(\zeta, t)|^2 S(f(\zeta, t)), \quad \text{in } U,\\ &\textit{Leibenson condition}\colon \Re\,(W) = 0, \quad \text{on } \partial U. \end{aligned} \tag{1.3}$$

Indeed, we have $\Delta_z W = \dfrac{1}{|f'(\zeta, t)|^2}\Delta_\zeta W$. In our study, one of the encountered difficulties will be the substitution in the source function S (which is a generalized function), from initial plane z used in $S(z)$ to the plane ζ of the unit disk in $S(\zeta)$.

Remark 1.5. In the particular case where $S(z) = \delta(z)$, we have

$$S(f(\zeta, t)) = \delta(f(\zeta, t)) = \frac{\delta(\zeta)}{|f'(0, t)|^2},$$

the term $|f'(0, t)|^2$ cancels out and the Stokes condition becomes

$$-\Delta_\zeta W = \delta(\zeta).$$

[1] The Eq. (1.2) is written without the real part usually used in the right hand side of the equality, but which is unnecessary. See the end of the proof.

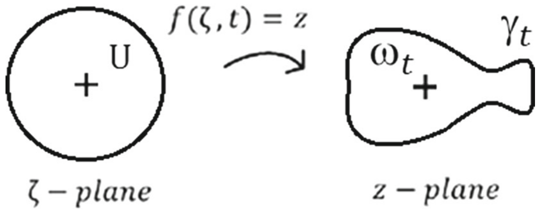

Fig. 1. Riemann mapping theorem

Most of the authors write the Stokes equation only with a delta Dirac function that is why the term $|f'|^2$ usually doe not appear.

A conformal map realized by the function f is represented on the Fig. 1.

Proof of Theorem 1.4. In ω_t, the flow is conservative due to the incompressibility ($\operatorname{div} \overrightarrow{V} = 0$ in $\Omega \backslash \operatorname{supp} S$) and irrotational because of relation $\overrightarrow{V} = -\nabla p$, so that we can introduce the complex velocity potential W (see [9]). Pressure p is then extended into the potential W in the complex plane [solution W of (1.3) is a multivalued function defined in $\omega_t \backslash \operatorname{supp} S$]. Function $f = f(\zeta, t)$ is the conformal transformation between the unit disk and the domain ω_t as each time t, as long as f remains univalent. We write

$$p(z,t) = \Re(W(z,t)) = \Re(W(f(\zeta,t),t).$$

We will use the same notation ξ for a 2-dimensional vector (ξ_1, ξ_2) and its corresponding complex number $\xi_1 + \imath\xi_2$.

Cauchy–Riemann equation takes the form

$$\frac{\partial W}{\partial z} = \frac{\partial p}{\partial x} - \imath \frac{\partial p}{\partial y},$$

knowing that on the boundary,

$$\overline{\frac{\partial W}{\partial z}} = \nabla p = -\dot{\mathbf{s}} = -(\dot{x} + \imath\dot{y}) = -\dot{z} = -\dot{f}(\zeta,t),$$

and that

$$\Re\left(\dot{\overline{f}} \frac{\partial f}{\partial \zeta}\zeta\right) = -\Re\left(\frac{\partial W}{\partial z}\frac{\partial f}{\partial \zeta}\zeta\right) = -\Re\left(\frac{\partial W}{\partial \zeta}\zeta\right),$$

we obtain the desired result because of $\dfrac{\partial W}{\partial \zeta}\zeta \in \mathbb{R}$ for $|\zeta| = 1$ as $\Re\left(W\Big|_{|\zeta|=1}\right) = 0.$ □

1.3. Polynomial Solution in Sink-Case

Let us briefly remind well-known results about explicit polynomial solutions in the Hele-Shaw problem with a unique sink (or source) point.

Substituting function S by a Dirac delta function at point $\mathbf{O}$ with strength -2π ($-\Delta p = -2\pi\delta_0$), we can easily solve the Dirichlet problem to obtain the following Polubarinova–Galin equation [3]

$$\Re(\dot{f}\ \overline{\zeta\ f'}) = -1, \quad |\zeta| = 1. \tag{1.4}$$

This configuration corresponds to a large range of applications in engineering, the obvious one being oil extraction. Notice that, when time increases, area of the domain decreases.

Let us introduce real polynomial functions of degree $N+1$ ($N \in \mathbb{N}^*$) (see [3])

$$f(\zeta,t)=\sum_{n=1}^{N+1} a_n(t)\zeta^n, \quad a_n(t)\in\mathbb{R}, \quad a_1(t)>0, \tag{1.5}$$

which can be injected in the Eq. (1.4), so that we finally obtain, by direct computation, the system of equations

$$\left\{\begin{pmatrix} a_1 & 2a_2 & \dots & Na_N & (N+1)a_{N+1} \\ 0 & a_1 & \dots & (N-1)a_{N-1} & Na_N \\ 0 & \dots & \dots & a_1 & 2a_2 \\ 0 & \dots & \dots & 0 & a_1 \end{pmatrix} + \begin{pmatrix} a_1 & 2a_2 & \dots & Na_N & (N+1)a_{N+1} \\ 2a_2 & 3a_3 & \dots & (N+1)a_{N+1} & 0 \\ Na_N & (N+1)a_{N+1} & 0 & \dots & 0 \\ (N+1)a_{N+1} & 0 & \dots & \dots & 0 \end{pmatrix}\right\} \begin{pmatrix} \dot{a_1} \\ \vdots \\ \vdots \\ \dot{a}_{N+1} \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Looking at the last equation of this system, we are able to integrate it and we get

$$a_1^{N+1}(t)a_{N+1}(t)=const. \tag{1.6}$$

Moreover, the first equation is the variation of the area of the fluid, which can be also integrated into a relation governing area evolution

$$a_1(t)^2+2a_2(t)^2+\dots+(N+1)a_{N+1}^2(t)=|\omega_0|-2t, \tag{1.7}$$

with $|\omega_0|$ being the initial area.

When time increases and gets closer to $\frac{|\omega_0|}{2}$ ($t\to\frac{|\omega_0|}{2}-0$), the Eq. (1.7) states that $a_1(t)$ and $a_{N+1}(t)$ tend to zero, when $a_1^{N+1}(t)a_{N+1}(t)$ should remain constant according to (1.6). This impossibility proves that the occuring of a cusp is unavoidable; at a moment $t^*<\frac{|\omega_0|}{2}$ the polynomial solution ceases to exist because of a lost of univalence (except in case $N=0$).

In this article, our purpose is to improve the capacity of extraction. We would like to get all the fluid, or at least the most we can. A first idea consists in the introduction of a second hole, allowing us to control the shape of the fluid and to increase the quantity of extracted fluid.

1.4. Sink/Source Configuration

Let us consider the following configuration. Within the fluid, on axis $O(x)$, are localized two holes, that is two Dirac delta functions with respective strengths Q_1 and Q_2. The first one of strength Q_1 is fixed on origin point **O** and the second one at a distance L from point **O**. The signs of the values Q_1 and Q_2 determine whether the hole is a source (>0) or a sink (<0). A function f realizes the conformal transformation between the unit disk and the fluid domain. Furthermore, the curve $f(\partial U,t)$ is symmetric relative to the x-axis. In these assumptions, knowing that $f([0,1],t)\in\mathbb{R}^+$ and that $f(1,t)$ belongs to the fluid boundary, we deduce the existence of a quantity $d(t)\in]0,1[$ depending on time and such that $f(d(t),t)=L$. As long as $d(t)$ remains less than 1, the hole takes place within the fluid and an equation of Polubarinova–Galin's type can be found. The geometrical situation of the transformation is presented on Fig. 2.

Proposition 1.6. *Polubarinova–Galin equation in sink/source configuration.*

If solving $f(d(t),t)=L$, we verify the inequality $d(t)<1$, then the shape of the boundary in the sink/source problem is governed by equation

$$\Re\left(\dot{f}\ \overline{\zeta\ f'}-\frac{Q_2}{2\pi}\frac{1+d(t)\zeta}{1-d(t)\zeta}\right)=\frac{Q_1}{2\pi}, \quad |\zeta|=1.$$

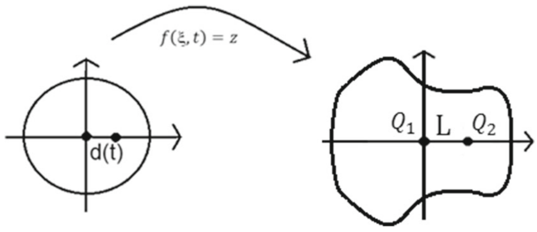

FIG. 2. Riemann mapping theorem in sink/source configuration

Proof. Substituting source term in the generalized Polubarinova–Galin equation by the function $S(\zeta) = Q_1\delta(\zeta) + Q_2\delta(\zeta - d)$ (see Remark 1.5), we obtain a solution for the Dirichlet problem, which can be explicitly described by elementary functions

$$W(\zeta, t) = -\frac{Q_1}{2\pi}\ln(\zeta) - \frac{Q_2}{2\pi}\ln\left(\frac{\zeta - d(t)}{d(t)\zeta - 1}\right),$$

and we obtain on $\partial U = \{\zeta | \zeta = e^{\imath\theta}, \theta \in [0; 2\pi[\}$,

$$-\frac{\partial W}{\partial \zeta}\zeta = \frac{Q_1}{2\pi} + \frac{Q_2}{2\pi}\frac{1 - d(t)^2}{1 - 2d(t)\cos\theta + d(t)^2}.$$

Quantity $\dfrac{1 - d^2}{1 - 2d\cos\theta + d^2}$ is called Poisson kernel. In particular, we have

$$\frac{1 - d^2}{1 - 2d\cos\theta + d^2} = \Re\left(\frac{1 + d\zeta}{1 - d\zeta}\right),$$

and the result is proved. □

As in the previous section, we would like to explicit a family of "simple" solutions, for example of polynomial type (1.5), in order that the resulting system of equations remains a finite-dimensional system from a optimal control point of view. However,

Proposition 1.7. *There is no polynomial solution to the Hele-Shaw problem in the sink/source configuration.*

Proof. Because of the relation $\Re\left(\dfrac{1 + d\zeta}{1 - d\zeta}\right) = \sum_{n=-\infty}^{+\infty} d^{|n|}\zeta^n$ we have

$$\Re\left(\frac{\partial f(e^{\imath\theta}, t)}{\partial t} e^{-\imath\theta}\overline{\frac{\partial f(e^{\imath\theta}, t)}{\partial \theta}}\right) = \frac{Q_1 + Q_2}{2\pi} + \frac{Q_2}{\pi}\sum_{n=1}^{+\infty} d^n \cos n\theta. \tag{1.8}$$

With a polynomial solution (1.5), we get, by identification of coefficients in front of cosine functions, $d(t)^k = 0, k \geq N + 1$. Then $d(t) = 0$ and the two Dirac functions should be on the same point **O**, which is impossible in the sink/source configuration. □

2. Multipole Configuration

Polynomial contours are close to real ones and should be studied in order to understand qualitative behavior of the problem. As we will see, there exists a way to control the domain while conserving the polynomial structure of the solutions; but this approach needs the introduction of pointwise configuration named *multipoles*, which was already introduced in [2]. We describe the building of multipoles and give the link between the classical approach by Richardson's moments and ours.

2.1. Dipole and Multipole Approximation

The limit calculations of this section can be avoided looking at the Sect. 1.1 (see Remark 2.2), nevertheless it seems to be useful to expose the physical building of multipoles.

To introduce a dipole, we consider that, in the unit disk, two holes with opposite strength Q are localized at a distance d and $-d$ from point **O** on the x-axis. We search the form of the solution when the two holes are getting closer (d tends to 0), while quantity $Q \cdot d$ remains constant (=D). We intentionally do not look at the real position of the holes in the z-plane.

We have to solve the following Dirichlet problem in the unit disk U (see Remark 1.5 for the removing of $|f'|^2$)

$$\begin{cases} -\Delta u_{(d)} = D\dfrac{\delta_d - \delta_{-d}}{2d}, & \text{in } U, \\ u_{(d)} = 0, & \text{on } \partial U, \end{cases}$$

and making d tend to 0. We notice that the right hand side of the Laplace equation tend to $D\,\delta_0'$.

We get

$$u_{(d)} = -\frac{D}{4\pi d}\ln\left|\frac{\zeta - d}{d\zeta - 1}\right| + \frac{D}{4\pi d}\ln\left|\frac{\zeta + d}{d\zeta + 1}\right|,$$

and if $\zeta = r\cos\theta + \imath r\sin\theta$, then

$$\begin{aligned} u_{(d)} = &-\frac{D}{8\pi d}\ln(d^2 + r^2 - 2dr\cos\theta) + \frac{D}{8\pi d}\ln(1 + d^2r^2 - 2dr\cos\theta) \\ &+\frac{D}{8\pi d}\ln(d^2 + r^2 + 2dr\cos\theta) - \frac{D}{8\pi d}\ln(1 + d^2r^2 + 2dr\cos\theta), \end{aligned}$$

when $d \to 0$ by taking pointwise limit outside the origin, we have

$$\begin{aligned} u_{(d)} \to u &= \frac{D}{8\pi}\left(\frac{4r\cos\theta}{r^2} - 4r\cos\theta\right) = \frac{D}{2\pi}\left(\frac{\cos\theta}{r} - r\cos\theta\right), \\ u &= \frac{D}{2\pi}\Re\left(\frac{1}{\zeta} - \zeta\right). \end{aligned}$$

Function $\zeta \mapsto \dfrac{1}{\zeta} - \zeta$ as a rational function is holomorphic on $\mathbb{C}^*$. Consequently, the complex solution of this Dirichlet problem when $d \to 0$ is

$$W(\zeta) = \frac{D}{2\pi}\left(\frac{1}{\zeta} - \zeta\right),$$

and

$$-\frac{\partial W}{\partial \zeta}\zeta = \frac{D}{\pi}\cos\theta, \quad |\zeta| = 1.$$

The infinite serie in (1.8) is reducted to a unique term, the identification can be done and we conserve the polynomial form of the solution.

To generate other control parameters by introducing high order multipoles, we add Dirac functions in the unit disk. An amount of $2n$ sources and sinks alternatively and regularly set on a small circle of radius d (see Fig. 3) leads to the following equations describing a multisource configuration

$$\begin{cases} -\Delta W_n = \dfrac{\pi}{2n\, d^n}\displaystyle\sum_{k=0}^{2n-1}(-1)^k\delta(d\,\mathrm{e}^{\frac{\imath k\pi}{n}}), & \text{in } U, \\ \Re\,(W_n) = 0, & \text{on } \partial U. \end{cases} \tag{2.1}$$

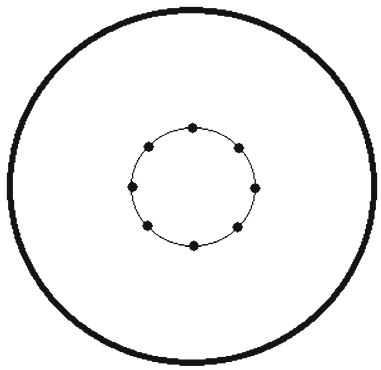

FIG. 3. A multisource configuration of degree 4

Remark 2.1. The right hand side of the Laplace equation in (2.1) tends to $\frac{(-1)^{n-1}\pi}{n!}\delta^{(n)}$ as $d \to 0$.

Without taking into account the boundary condition, we solve (2.1) in the whole complex plane and we get

$$W_n(\zeta) = \sum_{k=0}^{2n-1} (-1)^{k+1} \frac{1}{4n\, d^n} \ln\left(\zeta - d\, \mathrm{e}^{\frac{\imath k \pi}{n}}\right).$$

Our purpose is to obtain the form of the solution when d gets closer to 0. In [11], we have the relation

$$\frac{1}{2n} \Re\left(\frac{1}{\zeta^n}\right) = \lim_{d \to 0} \sum_{k=0}^{2n-1} (-1)^{k+1} \frac{1}{4n\, d^n} \ln\left|\zeta - d\, \mathrm{e}^{\frac{\imath k \pi}{n}}\right|.$$

Finally, the solution of the problem (2.1) when d tends to 0 is

$$W_n = \frac{1}{2n}\left(\frac{1}{\zeta^n} - \zeta^n\right), \tag{2.2}$$

noticing that a polynomial is a harmonic function and that the boundary condition is verified.

We now can combine different multipoles in a same point, considering that the harmonic function W in the unit disk has a growth rate at point $\mathbf{O}$ of kind

$$W(\zeta) = \ln(\zeta) + \frac{u_1}{2}\left(\frac{1}{\zeta}\right) + \cdots + \frac{u_N}{2N}\left(\frac{1}{\zeta^N}\right) + O(1), \tag{2.3}$$

with real u_i, $i = 1, \ldots, N$, being the strength of the ith multipole, and we solve the two equations

$$\begin{aligned} -\Delta W &= 0, \quad \text{in } U \backslash \{0\}, \\ \Re(W) &= 0, \quad \text{on } \partial U. \end{aligned} \tag{2.4}$$

If f is a polynomial solution of kind (1.5), it leads to a system of $(N+1)$ equations with N control parameters

$$\left\{\begin{pmatrix} a_1 & 2a_2 & \dots & Na_N & (N+1)a_{N+1} \\ 0 & a_1 & \dots & (N-1)a_{N-1} & Na_N \\ 0 & \dots & \dots & a_1 & 2a_2 \\ 0 & \dots & \dots & 0 & a_1 \end{pmatrix} + \begin{pmatrix} a_1 & 2a_2 & \dots & Na_N & (N+1)a_{N+1} \\ 2a_2 & 3a_3 & \dots & (N+1)a_{N+1} & 0 \\ Na_N & (N+1)a_{N+1} & 0 & \dots & 0 \\ (N+1)a_{N+1} & 0 & \dots & \dots & 0 \end{pmatrix}\right\} \begin{pmatrix} \dot{a}_1 \\ \vdots \\ \vdots \\ \dot{a}_{N+1} \end{pmatrix} = \begin{pmatrix} -2 \\ u_1 \\ \vdots \\ u_N \end{pmatrix}. \tag{2.5}$$

Remark 2.2. We a posteriori notice that the form of a multipole solution corresponds to the solution of the Laplace's equation with derivatives of the Dirac delta function in the right hand side. This result is justify by the Theorem 1.1.

Indeed, with $|f'^2(0,t)|S(\zeta) = \delta^{(n)}(\zeta)$, a solution of the Stokes condition is smooth outside the origin (see [5]) and coincides with

$$\frac{1}{2\pi}\ln^{(n)}(\zeta) = \frac{(-1)^{n-1}}{2\pi}(n-1)!\,\zeta^{-n}.$$

A unique solution due to restriction $\Re W|_{|\zeta|=1} = 0$ is (outside the origin)

$$W(\zeta) = \frac{(-1)^{n-1}}{2\pi}(n-1)!\left(\frac{1}{\zeta^n} - \zeta^n\right),$$

which gives the solution (2.2) by linearity and Remark 2.1.

2.2. Richardson's Moments

The previous method leads to a simple system of equations (2.5), however the physical understanding of these multipoles is not clear. Multipoles were added in the ζ-plane, but one can only act on the real world of the z-plane. This second approach needs the introduction of the system of Richardson's moments [13]. Contrary to Polubarinova–Galin equation, this system has a strong deficiency. Namely, the domain can not be in general reconstructed by its Richardson's moments. There is exceptions when one can guarantee the uniqueness of the domain with given Richardson's moments (see [16]). In other words, there are many different domains with identical Richardson's moments (see [14]) and this approach has to be used carefully.

We consider that a combination of multipoles takes place at point $\mathbf{O}$, so that the pressure at this point possesses the following growth rate

$$p(z) = \ln|z| + \frac{v_1}{2}\Re\left(\frac{1}{z}\right) + \dots + \frac{v_N}{2N}\Re\left(\frac{1}{z^N}\right) + O(1), \tag{2.6}$$

with real v_i, $i = 1, \dots, N$, being the strength of the ith multipole in the z-plane.

We define the Richardson's moments as follows

$$M_n = \frac{1}{\pi}\int_{\omega_t} z^n \mathrm{d}\Sigma, \quad n \in \mathbb{Z}.$$

Proposition 2.3. *With the combination of multipoles (2.6) at point* $\mathbf{O}$*, we have an infinite number of differential equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}M_0 = -2, \quad \frac{\mathrm{d}}{\mathrm{d}t}M_1 = v_1, \dots, \quad \frac{\mathrm{d}}{\mathrm{d}t}M_N = v_N, \quad \frac{\mathrm{d}}{\mathrm{d}t}M_k = 0, \quad k \geq N+1.$$

Proof. Using the Reynolds transport theorem and then the Green's formula, we set

$$\frac{\mathrm{d}}{\mathrm{d}t} M_n = \frac{1}{\pi} \int_{\gamma_t} z^n \, \dot{s} \cdot \nu \, \mathrm{d}s = -\frac{1}{\pi} \int_{\gamma_t} z^n \, \partial_\nu p \, \mathrm{d}s = \frac{1}{\pi} \int_{\omega_t} z^n (-\Delta p) \, \mathrm{d}\Sigma. \tag{2.7}$$

Looking at the previous Remark 2.2 (translated to the z-plane), we have the possibility to use Theorem 1.1 with source term

$$-\Delta p = S(z) = 2\pi\delta_0 + \sum_{k=1}^{N} v_k \frac{(-1)^k}{k!} \delta^{(k)}(z).$$

Substituting $-\Delta p$ by S in (2.7), we obtain the desired result. □

A very elegant result exists about Richardson's moments in the polygonal case.
With the map $f(\zeta, t) = z$ from the unit disk U we have

$$M_n = \frac{1}{2\imath\pi} \int_{\partial U} f^n(\zeta, t) f' \overline{f} \mathrm{d}\zeta.$$

With a polynomial (1.5) of degree $N + 1$ we get N+1 relations (see [7]) by residue theorem

$$M_n = \sum_{i_1, \dots, i_{n+1}} i_1 a_{i_1}(t) a_{i_2}(t) \dots a_{i_{n+1}}(t) a_{i_1 + \dots + i_{n+1}}(t), \quad n \in \{0, \dots, N\}.$$

In this case, the system of equations is reduced to a finite system of $N + 1$ equations because of $M_k = 0$, $k \geq N + 1$.

2.3. Links Between Richardson's Moments and Polubarinova–Galin Equation

We found two different ways to control the system. Control parameters u_n can be introduced in the Polubarinova–Galin equation and we obtain a simple system of quadratic equations but the interpretation of the control parameters is not clear. The second approach used the Richardson's moments with control parameters v_n. In this case, the physical interpretation of the multipole is easy to make; however we work with moments, which are complicated polynomials of high degrees and we can not establish how the coefficients of the polynomial solution (1.5) change during time evolution.

When the source term $S = S(z)$ in the Stokes equation of the real plane is made of a combination of derivatives of delta Dirac functions, the susbitution of z by ζ is difficult to make but it gives the links between Polubarinova–Galin equation and the Richardson's moments. We write

$$\Delta_z u = -2\pi\delta(z) + \frac{(-1)^1}{1!} v_1 \delta^{(1)}(z) + \dots + \frac{(-1)^N}{N!} v_N \delta^{(N)}(z)$$

and we search the corresponding coefficients $v_i = U_i(u_1, \dots, u_N)$ after substitution by ζ in the equation

$$\Delta_\zeta u = -2\pi\delta(\zeta) + \frac{(-1)^1}{1!} u_1 \delta^{(1)}(\zeta) + \dots + \frac{(-1)^N}{N!} u_N \delta^{(N)}(\zeta).$$

The following theorem describes these links, which exist between the control parameters u_n and v_n for a univalent function from the unit disk to the current domain (not only for a polynomial solution!)

Theorem 2.4. *When considering a univalent function f from the unit disk to the considering domain, links between the control parameters $u = (u_1, \dots, u_N)$ from the Polubarinova–Galin equation and the second ones $v = (v_1, \dots, v_N)$ used in the Richardson's moments are described by relations*

$$\frac{v_n}{n} = \sum_{i_1, \dots, i_n} i_1 \left(\frac{f^{(i_1)}(0)}{i_1!} \right) \left(\frac{f^{(i_2)}(0)}{i_2!} \right) \dots \left(\frac{f^{(i_n)}(0)}{i_n!} \right) \frac{u_{i_1 + \dots + i_n}}{i_1 + \dots + i_n}, \quad n \in \{1, \dots, N\}. \tag{2.8}$$

Proof. Let us fix a moment $t = t_0$ and write $f(\zeta) = f(\zeta, t_0)$ for short. From the relation $\delta(f(\zeta)) = \frac{\delta(\zeta)}{|f'(t_0,t)|^2}$, we use the Fa di Bruno's formula to write

$$\frac{\delta^{(n)}(\zeta)}{|f'(t_0,t)|^2} = \sum_{\sum ik_i=n} \binom{n}{k_1,\ldots,k_j} \delta^{(k_1+\cdots+k_n)}(z) \left(\frac{f^{(1)}(\zeta)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(\zeta)}{n!}\right)^{k_n}. \tag{2.9}$$

However, δ function and its derivatives are generalized functions and should be considered under the action of a test function $\phi \in D$.

For an entire $n \in 1, \ldots, N$,

$$\frac{(-1)^n u_n}{n!}\langle \delta^{(n)}, \phi\rangle = \int_{\mathbb{C}} \frac{(-1)^n}{n!} u_n \delta^{(n)}(\zeta)\phi(\zeta)\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} = \int_{\mathbb{C}} \frac{u_n}{n!}\delta(\zeta)\phi^{(n)}(\zeta)\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

Noticing that

$$\int_{\mathbb{C}} \frac{u_n}{n!}\delta(z)\phi^{(n)}(z)\mathrm{d}z \wedge \mathrm{d}\overline{z} = \int_{\mathbb{C}} \frac{u_n}{n!}\delta(f(\zeta))\phi^{(n)}(f(\zeta))|f'(\zeta)|^2\mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta},$$

we apply the relation (2.9) to the function ϕ insteed of δ and we obtain a relation between control parameters u_i and v_i of the following form

$$\frac{v_n}{n!} = \sum_{j=1}^{N} \sum_{\sum k_i=n, \sum ik_i=j} \binom{j}{k_1,\ldots,k_j} \left(\frac{f^{(1)}(0)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(0)}{n!}\right)^{k_n} \frac{u_j}{j!},$$

which can be simplified into

$$\frac{v_n}{n} = \sum_{j=1}^{N} \sum_{\sum k_i=n, \sum ik_i=j} \binom{n-1}{k_1,\ldots,k_j} j \left(\frac{f^{(1)}(0)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(0)}{n!}\right)^{k_n} \frac{u_j}{j}.$$

Moreover, with the formula (comparison of the derivative in the multinomial theorem),

$$\begin{aligned} &\sum_{\sum k_i=n, \sum ik_i=j} \binom{n-1}{k_1,\ldots,k_j} j \left(\frac{f^{(1)}(0)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(0)}{n!}\right)^{k_n} \\ &\quad = \sum_{i_1,\ldots,i_n=j} i_1 \left(\frac{f^{(i_1)}(0)}{i_1!}\right)\left(\frac{f^{(i_2)}(0)}{i_2!}\right) \cdots \left(\frac{f^{(i_n)}(0)}{i_n!}\right) \end{aligned}$$

the theorem is established. $\square$

Remark 2.5. In particular, we have, for a polynomial solution of type (1.5)

$$v_1 = a_1 u_1 + \cdots + a_N u_N,$$

and

$$v_N = a_1^N u_N.$$

An example with $N = 3$. We have

$$\begin{aligned} v_1 &= a_1 u_1 + a_2 u_2 + a_3 u_3, \\ v_2 &= a_1^2 u_2 + 2a_1 a_2 u_3, \\ v_3 &= a_1^3 u_3. \end{aligned}$$

3. Study of the Controlability

3.1. Sufficient Conditions

We are looking for conditions on control parameters to make the system controllable. We chose to work with the u-parameters from the Polubarinova–Galin equation; we will justify this approach.

Theorem 3.1. *For any polynomial solutions (1.5) of degree $N+1$, we are able to extract all the fluid while contour γ_t of the spot remains smooth, using the multipole structure with the following restrictions*

$$\begin{cases} |u_n| + |u_{N+1-n}| \le 2, & n = 1, \dots, E[N/2], \\ |u_n| \le 1, & n = E[N/2]+1, \dots, N, \end{cases}$$

with $E[.]$ being the floor function.

Proof. To extract all the fluid, we need to find a particular solution of the problem, which exists until the whole absorption of the fluid (until $\sum i a_i^2 > 0$). Let us design a control $u_k(t)$ such that the contour γ_t verifies the equalities

$$a_i(t) = c_i a_1(t)^i, \quad i = 1, \dots, N+1,$$

where c_i are constants and can be found by relations

$$c_i = \frac{a_i(0)}{a_1(0)^i}.$$

We obtain a corresponding polynomial function f of the form

$$f(\zeta, t) = \sum_{n=1}^{n=N+1} c_i (a_1(t)\zeta)^n.$$

We claim that, if the initial solution $f(\zeta, 0)$ is univalent and if $a_1(t)$ is decreasing, then $f(\zeta, t)$ is univalent until the end.

Indeed, $f(\zeta, t) = f\left(\frac{a_1(t)}{a_1(0)}\zeta, 0\right)$.

Using $0 < a_1(t) \le a_1(0)$, we obtain that $f(\zeta, t)$ coincides with $f(\zeta, 0)$ on the subdisk $|\zeta| \le \frac{a_1(t)}{a_1(0)} \le 1$.

Knowing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a_i}{a_1^i}\right) = 0,$$

that is

$$\dot{a_i} = i a_i \frac{\dot{a_1}}{a_1},$$

we rewrite the system of equations (2.5) in another form

$$\begin{cases} \left(\sum_{i=1}^{N+1} i^2 a_i^2\right) \frac{\dot{a_1}}{a_1} = -1\,, \\ \left(\sum_{i=1}^{N+1-n} 2i(i+n) a_i a_{i+n}\right) \frac{\dot{a_1}}{a_1} = u_n, \quad n = 1, \dots, N. \end{cases}$$

First equation states that a_1 is decreasing.

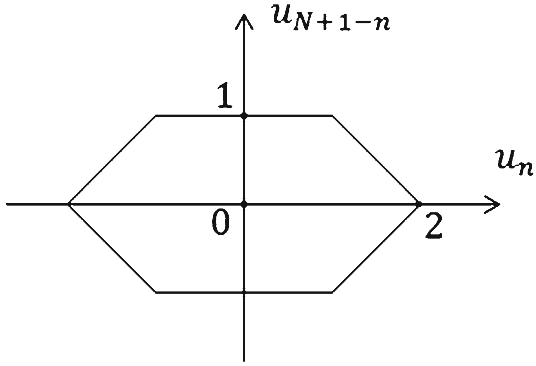

FIG. 4. Domain of controlability in the plane (u_n, u_{N+1-n})

By elementary calculations, using the relation $2ab \leq a^2 + b^2$, we notice that for $n > E[N/2]$,

$$\left| \sum_{i=1}^{N+1-n} 2i(i+n)a_i a_{i+n} \right| \leq \sum_{i=1}^{N+1} i^2 a_i^2,$$

i.e.,

$$|u_n| \leq 1.$$

For $n \leq E[N/2]$,

$$\left| \sum_{i=1}^{N+1-n} 2i(i+n)a_i a_{i+n} \right| + \left| \sum_{i=1}^{n} 2i(i+N+1-n)a_i a_{i+N+1-n} \right| \leq 2 \sum_{i=1}^{N+1} i^2 a_i^2,$$

i.e.,

$$|u_n| + |u_{N+1-n}| \leq 2.$$

□

In the planes (u_n, u_{N+1-n}), $n = 1, \ldots, E[N/2]$ the domain of controlability takes the form of an hexagon, as represented on Fig. 4.

In the case of the v-parameters in z-plane the situation with controllability is much more complicated and we will further see in a simple example in dimension 2 that the boundaries for the control have to be considered on the u-control parameters in order to have a result not depending on the coefficients of the polynomial to obtain controllability.

3.2. Necessary Conditions

In the next section, we will see that, for the 2-dimensional case, $fa_1\zeta + a_N\zeta^N$, $N > 1$, the condition $|u_N| \leq 1$ is necessary for controllability. The dimension 3 gives us a similar result and we can guess the following proposition.

Let $U \subseteq \mathbb{R}^N$ denote the chosen closed convex set of admissible controls, i.e. we assume $(u_1, \ldots, u_N) \in U$.

Conjecture 3.2. *For a polynomial solution (1.5) of degree $N+1$, if there exists an admissible control extracting all the fluid for any initial spot, then U contains the set*

$$\sum_{n=1}^{N} |u_n| \leq 1.$$

4. Optimal Control in Dimension 2

4.1. Equations of the Problem

In dimension 2, the considered polynomials are of the form $f(\zeta,t) = a_1(t)\zeta + a_N(t)\zeta^N$ with $N \geq 2$. The system of equations becomes

$$\begin{pmatrix} a_1 & Na_N \\ Na_N & a_1 \end{pmatrix} \begin{pmatrix} \dot{a_1} \\ \dot{a_N} \end{pmatrix} = \begin{pmatrix} -1 \\ u \end{pmatrix},$$

where $u = u_{N-1}$ is the unique control parameter.

We obtain

$$\begin{pmatrix} \dot{a_1} \\ \dot{a_N} \end{pmatrix} = \frac{1}{a_1^2 - N^2 a_N^2} \begin{pmatrix} -a_1 - uNa_N \\ Na_N + ua_1 \end{pmatrix}. \tag{4.1}$$

4.2. Necessary and Sufficient Condition of Controlability

Let us assume that u belongs to a closed convex set $U \subset \mathbb{R}$.

Proposition 4.1. *The necessary and sufficient condition of controlability in the 2-dimensional case is*

$$|u| \leq 1.$$

Proof. The sufficient condition of controlability obtained in Theorem 3.1 can be easily translated to this particuliar case and expressed by the inequality

$$|u| \leq 1. \tag{4.2}$$

We already know that this condition is sufficient, we prove that it is also necessary. By introduction of the unknown X such that $X = \dfrac{Na_N}{a_1}$, we obtain the equation

$$\dot{X} = \frac{1}{a_1^2}\frac{1}{1-X^2}[uX^2 + X(N+1) + uN].$$

Binomial $a_1(t)\zeta + a_N(t)\zeta^N$ remains univalent until $|X| = |\dfrac{Na_N}{a_1}| < 1$.

Rescaling the time (conserving its orientation), we simplify the expression into

$$\dot{X} = uX^2 + X(N+1) + uN = g(X,u).$$

For $X \in [-1,1]$ in a neighborhood of points $X = 1$ and $X = -1$, the velocity $\dot{X}$ should be oriented toward the point $X = 0$.

If these conditions are verified then

$$\begin{cases} \exists u \,|\, g(1,u) \leq 0, \\ \exists u \,|\, g(-1,u) \geq 0, \end{cases} \tag{4.3}$$

consequently

$$\begin{cases} \exists u \,|\, (N+1)(u+1) \leq 0 \Leftrightarrow u \leq -1, \\ \exists u \,|\, (N+1)(u-1) \geq 0 \Leftrightarrow u \geq 1. \end{cases}$$

Then, we have a domain of univalence necessary containing interval $[-1,1]$ in order to verify the conditions (4.3), so that the equality (4.2) is a necessary and sufficient condition for the controlability of the system in dimension 2. □

If the control is taken in the z-plane, we have the relation $v = a_1^N u$ (see Theorem 2.4) and the boundary of the domain of controllability for v depends on coefficient a_1. For a bigger a_1, we will need a bigger v, even if the geometry is the same (homothetic). The optimality in Pontryagin maximum principle imposes to have $v = const$ for a certain interval of time while a_1 decreases. From an engineer point of view, we will need a constant high power, whereas if we are able to estimate a_1 during time evolution, the used power can be chosen smaller for the same result (absorption of all the fluid).

4.3. Optimal Control

We would like to minimize a quantity

$$\int_{t=0}^{t=T} h(a_1, a_N)\mathrm{d}t,$$

during time evolution of the system. T is fixed and corresponds to the maximal time of existence of the spot in sink-case. Our aim is to get closer to the circle in order to avoid the occuring of singularities, that's why the function h will be chosen to be minimal for the circle. We will work with three main examples, $h(a_1, a_N) = a_N^2$, $h(a_1, a_N) = \dfrac{a_N^2}{a_1^2}$ and $h(a_1, a_N) = (\dfrac{a_N}{a_1^i})^2$.

We apply the Pontryagin maximum principle. Hamiltonian H is the following one

$$H = (p \quad q)\begin{pmatrix} \dot{a_1} \\ \dot{a_N} \end{pmatrix} - h(a_1, a_N) \quad = \frac{1}{a_1^2 - N^2 a_N^2}\left[-a_1 p + u(a_1 q - N a_N p) + N a_N q\right] - h(a_1, a_N).$$

If $a_1 q - N a_N p > 0$, then $u = u_{max} = 1$, we have $\dot{a_1} + \dot{a_N} = 0$.

If $a_1 q - N a_N p < 0$, then $u = u_{min} = -1$, we have $\dot{a_1} - \dot{a_N} = 0$.

If $a_1 q - N a_N p = 0$, then u can not be obtained from the maximum condition (it leads to singular extremals).

Optimal trajectories are straights with derivative ± 1 in the space phases (a_1, a_N) (see Fig. 5).

Now, we want to construct optimal synthesis. Singular extremals usually play a key role in investigating behaviour of optimal synthesis. So, let us find singular extremals in this problem, by solving

$$a_1 q = N a_N p, \quad t \in (t_1, t_2), \tag{4.4}$$

for some $t_1, t_2 \in \mathbb{R}$.

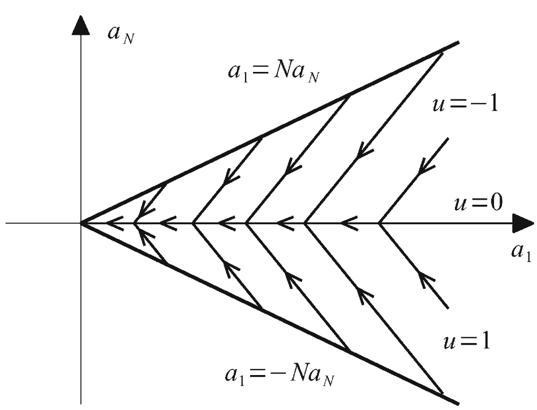

Fig. 5. Optimal trajectories in space phases

We know that

$$-\dot{p} = \frac{p + uq}{a_1^2 - N^2 a_N^2} - h'_{a_1},$$
$$-\dot{q} = N\frac{-q - up}{a_1^2 - N^2 a_N^2} - h'_{a_N}.$$

Differentiating (4.4) and using (4.1), we have the equation

$$a_1 h'_{a_N} = N a_N h'_{a_1}.$$

With our examples,

if $h(a_1, a_N) = a_N^2$, then $2a_N a_1 = 0$, and $a_N = 0$;

if $h(a_1, a_N) = \dfrac{a_N^2}{a_1^2}$, then $\dfrac{2a_N}{a_1}(1 + \dfrac{Na_N^2}{a_1^2}) = 0$ and $a_N = 0$;

if $h(a_1, a_N) = (\dfrac{a_N}{a_1^i})^2$, then $a_N(a_1^2 + N^2 a_N^2) = 0$, and $a_N = 0$.

In all the proposed examples, we have $a_N = 0$, so that $\dot{a_N} = 0$, i.e. $u = 0$ because of $a_1 \neq 0$ and $a_N = 0$.

Proposition 4.2. *In the particular case of a binomial solution f, the contour is optimally controllable (with chosen functions h) until total aspiration of the fluid with the choice of the following control parameter:*

(i) $u = 1$ $(a_N < 0)$ *or* $u = -1$ $(a_N > 0)$ *until the occuring of a circular contour;*

(ii) $u = 0$ *from the moment when the contour became a circle.*

Remark 4.3. Optimality can be easily proved be Zelikina's theorem (see [17]).

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