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ATTRACTORS–REPELLERS IN THE SPACE OF CONTOURS IN THE STOKES–LEIBENSON PROBLEM FOR HELE–SHAW FLOWS

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It is shown that, in the space of quasicontours playing a role of free boundaries in the Stokes–Leibenson problem, there is a manifold of codimension 1 *such that some points of this manifold are attractors in the case of sink and repellers in the case of source, whereas, on the contrary, other points are repellers in the case of sink and attractor in the case of source. Bibliography*: 36 *titles. Illustrations*: 7 *figures.*

1 The Stokes–Leibenson Problem. Solvability Theorems. Perturbation of a Circle

1.1. We recall the classical setting of the Stokes–Leibenson problem. Let Ω_0 be a simply connected domain in $\mathbb{R}^2$ bounded by a sufficiently smooth curve Γ_0 surrounding the origin $\{0\}$. The domain Ω_0 is interpreted as a spot of liquid at some initial time with a source/sink located at the origin $\{0\} \in \Omega_0$. The domain is deformed as follows: at time t, a point $\mathbf{s}(t) = \big(x(t), y(t)\big)$ on the boundary Γ_t of Ω_t moves with the velocity $\dot{\mathbf{s}} = (\dot{x}, \dot{y})$ determined by the kinematic condition

$$\dot{\mathbf{s}} = \nabla u \quad \text{on} \quad \Gamma_t, \tag{1.1}$$

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where $\nabla u = (u_x, u_y)$ is the gradient of a function $u(t;\cdot,\cdot) : \Omega_t \to \mathbb{R}$ satisfying the Stokes condition [1]

$$u_{xx} + u_{yy} = q\delta(x,y) \ \text{ in } \ \Omega_t \tag{1.2}$$

and the dynamic Leibenson condition [2]

$$u = 0 \ \text{ on } \ \Gamma_t, \tag{1.3}$$

where $\delta(x,y)$ is the Dirac δ-function concentrated at the origin and the nonzero coefficient $q \in \mathbb{R}$ characterizes the strength of source/sink. Let Ω_t be symmetric with respect to the x-axis. It is convenient to assume that $q = 2$. Moreover, $t > 0$ corresponds to the case of source, whereas $t < 0$ is related to the case of sink.

We recall results of the study of the problem (1.1)–(1.3) coming back to works of Galin [3] and Kochina [4, 5] published in 1945. In the mentioned papers, the following equation was obtained

$$2\pi \operatorname{Re}\left[\, \dot{f}(\zeta,t)\,\overline{\zeta f'(\zeta,t)}\,\right] = q, \quad |\zeta| = 1, \tag{1.4}$$

for a univalent mapping $f(\cdot,t)$ from the unit disk $\{\zeta \in \mathbb{C} \,\big|\, |\zeta| \leqslant 1\}$ to the original domain Ω_t. Under the assumption that $f(\zeta,t) = a_1(t) + a_2(t)\zeta + \ldots + a_n(t)\zeta^n$, a system of ordinary differential equations for the coefficients $a_j(t)$ was derived in [3]. For $n = 2$ the system takes the form [5]

$$\begin{aligned} &a_1^2(t)a_2(t) = a_1^2(0)a_2(0),\\ &a_1^2(t) + 2a_2^2(t) = a_1^2(0) + 2a_2^2(0) - qt/\pi. \end{aligned}$$

If $|a_2/a_1| < 1/2$, then the image of the unit circle under the mapping $f(\cdot,t)$ is a Pascal snail Γ_0 with the polar radius $\{r(\theta) = a\cos\theta + b, b > a\}$. In the case of sink (i.e., for $qt < 0$), it is transformed in a finite time to a cardioid Γ_{t_*} whose vertex turns out does not reach the sink point. Then (more exactly, for $qt < qt_*$ when $|a_2/a_1| > 1/2$) the mapping $f(\cdot,t)$ fails to be univalent and the solution $t \mapsto \Gamma_t$ fails to exist. Kufarev [6] observed a similar effect for a circle with center shifted relative to the sink point. The result of Kufarev was proved 24 years later by Richardson [7] on the basis of his theorem [7] asserting the constancy of complex moments

$$M_n \overset{def}{=} \int\limits_{\Omega_t} z^n\, \overline{dz} \wedge dz, \quad n \geqslant 1.$$

The first results on the local solvability of (1.4) were established in [8].

In [3]–[5] and [8], the problem (1.1)–(1.3) was interpreted as a flow of a viscous fluid through a porous medium section. In the last decades, such problems were also referred to as Hele–Shaw flow problems or free boundary Hele–Shaw problems . The growing interest in such problems (cf., in particular, [9]–[14] and the references therein) is caused by not only its applications to engineering practice, materials science, processes related to crystal growth (cf., for example, [15]–[17]), but also the fact that such problems turn out to be good models (cf., for example, [18]–[20]) of rather complicated two-phase problems. Hence the study of such models can be helpful for predicting singularities of the leading terms of asymptotics of solutions to phase transition problems (cf., in particular, [21]).

1.2. An important point in the study of the problem (1.1)–(1.3) is understanding of the unexpected fact that, in the case of sink ($qt < 0$), the solution does not exist for any small t if the initial contour Γ_0 is not analytic at least at one point. In other words, the case of sink is impossible for such an initial contour: the problem has no solution. The reason of this effect is explained below. The question arises whether such a (nonanalytic) contour may be shifted in the case of source? An answer to this question was recently obtained in [22, 23], where it was proved that for small times the solution to the problem (1.1)–(1.3) exists in the case of source. At the first glance, it is surprising since the law of motion of the contour Γ_0 is *a priori* independent of its analyticity and the type of "motor," source or sink. The internal reason of these facts is clarified below, when we consider an evolution of harmonics of weak perturbations of a circle.

In $\Omega_t^+ = \Omega_t \cap \mathbb{R}^2_+$, we introduce the harmonic conjugate v of u and parametrize a pont $\mathbf{s}(t) \in \Gamma_t^+ = \Gamma_t \cap \mathbb{R}^2_+$ by the number η determining the level curve $\{(x,y) \in \Omega_t \big| v(t;x,y) = \eta\}$ of the function v which contains the point $\mathbf{s}(t)$. By the symmetry of Ω_t and the (appropriate) choice of the value $q = 2$, the parameter η varies from 0 to 1. Thereby we have the continuous function

$$s(t,\cdot) : [0,1] \ni \eta \mapsto s(t,\eta) = |s_0 s_\eta| \in [0, |\Gamma_t^+|],$$

where $|s_0 s_\eta|$ is the length of arc $\smile{s_0 s_\eta}$ of the curve Γ_t^+, measured in the positive direction from the intersection point s_0 of the curve Γ_t^+ and the positive half x-axis to the intersection point s_η of this curve and the level curve $\{v(t;x,y) = \eta\}$.

For every t, in the closure of the half-axis

$$\Pi = \left\{u + iv \in \mathbb{C} \big| -\infty < u < 0,\ 0 < v < 1\right\},$$

we define the Helmholtz–Kirchhoff function [24] by the formula

$$A + iB : w = u + iv \mapsto A(t;u,v) + iB(t;u,v) = \ln\left|\frac{\partial z(t;w)}{\partial w}\right| + i \arg \frac{\partial z(t;w)}{\partial w}, \tag{1.5}$$

where $z = x + iy \in \Omega_t^+$. We have

$$\begin{aligned} x(t,v) &= x(t,0) - \int_0^v e^{a(t,\eta)} \sin b(t,\eta)\, d\eta, \\ y(t,v) &= \int_0^v e^{a(t,\eta)} \cos b(t,\eta)\, d\eta, \end{aligned} \tag{1.6}$$

where

$$x(t,0) = \int_{-\infty}^0 e^{A(t;u,0)}\, du$$

amd

$$a(t,v) + ib(t,v) = A(t;0,v) + iB(t,0,v). \tag{1.7}$$

Since

$$|\nabla u|\big|_{s=s(t,v)\in\Gamma_t} = e^{-a(t,v)},$$

the kinetic condition (1.1) can be written as

$$e^{-a} = \dot{x}\cos b + \dot{y}\sin b. \tag{1.8}$$

Thus, the problem (1.1)–(1.3) can be formulated in terms of a and b. Namely [25], the problem (1.1)–(1.3) is equivalent to the Cauchy problem for the following nonlinear integro-differential equation:

$$\dot{b}(t,v) = e^{-2a}a' + b'e^{-a}\int\limits_0^v \left[e^a\dot{a} - e^{-a}b'\right] d\xi, \quad \dot{b} \overset{def}{=} \frac{\partial b}{\partial t}, \quad b' \overset{def}{=} \frac{\partial b}{\partial v} \tag{1.9}$$

with respect to the functions a and b, connected (for every t) by the Hilbert transform: by (1.7), the functions a and b are the trace for $u = 0$ of harmonic conjugate functions A and B in the half-strip Q. In fact, this is an obstacle for the existence of a solution in the case of sink provided that Γ_0 is not analytic at least at one point. The matter is that Equation (1.9) hardly connects the possibility of evolution of the function $t \to b(t,v)$, i.e., the evolution of the angle of slope of the tangent to Γ_t at the point $s_v \in \Gamma_t$, with the evolution of the coefficient $\exp a(t,v)$ of longitudinal deformations of the contour at this point. In the case of sink (i.e., for $t < 0$), where the contour "shrivel," the longitudinal deformation of the contour at the point $s_v \in \Gamma_0$, determined by Equation (1.9), is impossible provided that the contour Γ_0 is not analytic at this point. In this case, Equation (1.9) (for any small $-t > 0$) is unsolvable because of the reason clarified by Theorem 1.2 and connected with the exponential decay of the Fourier coefficients of an analytic function on the circle.

1.3. We note that $B(t,u,v) \equiv \pi v$ if (and only if) the curve Γ is a circle $\{z \in \mathbb{C} \big| |z| = R_0 > 0\}$ with center at the origin (i.e., the support of the δ-function in Equation (1.2)). Taking into account this remark, we represent b in the form

$$b(t,v) = \pi v + \beta(t,v).$$

We note that $\beta(t,0) = \beta(t,1) = 0$ by the symmetry and differentiability of Γ, and expand the function $\beta(t,\cdot) : [0,1] \ni v \mapsto \beta(t,v)$ into the Fourier series

$$\beta(t,v) = \sum_{k \geqslant 1} \beta_k(t) \sin \pi k v$$

relative to the orthogonal basis $e_k : [0,1] \ni v \mapsto \sin \pi k v$, $k \in \mathbf{N}$, for the space $L^2(0,1)$.

It is easy to see that the same Fourier coefficients $\beta_k(t)$ determine a function $\alpha_0(t)$ given by the equality

$$A(t;u,v) = \alpha_0(t) + \pi u + \sum_{k \geqslant 1} \beta_k(t) e^{\pi k u} \cos \pi k v.$$

It is clear that it can be also expressed by the equality

$$a(t,v) = \alpha_0(t) + \alpha(t,v), \quad \alpha(t,v) = \sum_{k \geqslant 1} \beta_k(t) \cos \pi k v.$$

Indeed, the increment of the area $|\Omega_t| = 2|\Omega_t^+|$ of Ω_t per time unit is equal to

$$\int\limits_\Gamma \dot{\mathbf{s}}\, d\Gamma \overset{(1.1)}{=} \int\limits_\Gamma \frac{\partial u}{\partial \nu}\, d\Gamma,$$

i.e., the coefficient at the δ-function in Equation (1.2). In other words,

$$\frac{d}{dt}|\Omega_t^+| = 1 \iff |\Omega_t^+| = t + t_0, \quad t_0 = |\Omega_0^+|. \tag{1.10}$$

In terms of the Helmholtz–Kirchhoff function, this condition is written as

$$\frac{d}{dt}\int_0^1 \Big(\int_{-\infty}^0 e^{2A(t;u,v)}\,du\Big)\,dv = 1$$

since the Jacobian of $w \mapsto e^{A+iB}$ is equal to $|\partial z/\partial w| \overset{(1.5)}{=} e^{2A}$. Hence the coefficient $e^{a(t,v)}$ of the longitudinal deformation of the contour Γ is found from Equation (1.10). Therefore, this coefficient, and, consequently, the function $\alpha_0(t)$, depends on $|\Omega_0|$ and the Fourier coefficients $\beta_k(t)$. Thus, the required deformation of the initial curve Γ_0, determined by the Cauchy problem for Equation (1.9), is completely characterized by the evolution of the Fourier coefficients $\beta_k(\cdot)$ in the expansion

$$\beta(t,v) = \sum_{k\geqslant 1}\beta_k(t)\sin\pi kv.$$

Theorem 1.1 (cf. [26]). *Equation* (1.9) *can be written in the form*

$$\dot\beta - \mathbf{K}(\beta)\dot\beta = \frac{1}{2(t+t_0)}\,\mathbf{F}(\beta), \tag{1.11}$$

where $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1$ *and* $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1$ *with*

$$\begin{aligned}
[\,\mathbf{K}_0(\beta)\dot\beta\,]\,(t,v) &= b'(t,v)\,e^{-\alpha(t,v)}\int_0^v e^{\alpha(t,\eta)}\,\dot\alpha(t,\eta)\,d\eta,\\
[\mathbf{K}_1(\beta)\dot\beta](t,v) &= \Big(\sum_{j\geqslant 1}\frac{2\beta_j\dot\beta_j}{j+1}\Big)\sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k}\sin\pi kv + [\mathbf{r}_1(\beta)\dot\beta](t,v),\\
[\mathbf{F}_0(\beta)](t,v) &= \frac{1}{\pi}\Big\{e^{-2\alpha}\alpha' - b'e^{-\alpha}\int_0^v e^{-\alpha}b'\,d\eta + \pi b'e^{-\alpha}\int_0^v e^{\alpha}\,d\eta\Big\},\\
[\mathbf{F}_1(\beta)](t,v) &= \Big(\sum_{j\geqslant 1}\frac{2\beta_j^2}{j+1}\Big)\sum_{k\geqslant 1}\frac{(-1)^{k-1}}{k}\sin\pi kv + [\,athbfs_1(\beta)](t,v).
\end{aligned}$$

Moreover,

$$|\mathbf{r}_1(\beta)\dot\beta| \leqslant C\|\beta\|_1^2\|\dot\beta\|_0, \quad |\mathbf{s}_1(\beta)| \leqslant C\|\beta\|_1^3,$$

where

$$\|\beta\|_1 = \max_t\sqrt{\sum_{k\geqslant 1}\Big(k\beta_k(t)\Big)^2}, \quad \|\dot\beta\|_0 = \max_t\sqrt{\sum_{k\geqslant 1}\Big(\dot\beta_k(t)\Big)^2}.$$

The dynamic system is written in the coordinate form as follows:

$$\begin{aligned}
2(t+t_0)(\beta_1\dot\beta_1 + r_1(\beta)\dot\beta) &= (-\beta_1^2 + 2\sum_{j\geqslant 2}\beta_j^2\Big) + s_1(\beta),\\
2(t+t_0)(\dot\beta_k + r_k(\beta)\dot\beta) &= -(k+2)\beta_k + s_k(\beta), \quad k\geqslant 2,
\end{aligned}$$

where $|r_k(\beta)\dot\beta| \leqslant C\|\beta\|_1^{1+\operatorname{sgn}|k-1|}\|\dot\beta\|_0$ *and* $|s_k(\beta)| \leqslant C\|\beta\|_1^{2+\operatorname{sgn}|k-1|}$.

The following assertion is proved by using Theorem 1.1 with the help of circumstantial functional-geometric constructions.

Theorem 1.2 (cf. [25]–[27])**.** *If the initial contour* Γ_0 *is a small* (*defined by* (1.2)) *perturbation of the circle of radius* $R(0)$*, then, in the case of source, the evolution* Γ_t *continues infinitely long and is it unique on any time-interval. Starting with some time, the deviation of* Γ_t *from the circle of radius* $R(t) = \sqrt{R^2(0) + 2t/\pi}$ *almost monotonically tends to zero. More exactly, there exists a number* $\rho \in (0, 1/8)$ *such that for any* $\mu \in (0, 1)$ *the function*

$$[0,1] \ni v \mapsto b(t,v) = \pi v + \beta(t,v), \quad \beta(t,v) = \sum_{n\geqslant 1} \beta_n(t) \sin \pi n v, \tag{1.12}$$

determining the contour Γ_t*, exists for any* $t > 0$ *only if the initial angular function determining the contour* Γ_0*, i.e., the function*

$$[0,1] \ni v \mapsto b(0,v) = \pi v + \sum_{n\geqslant 1} \beta_n^0 \sin \pi n v, \tag{1.13}$$

satisfies the condition

$$\sum_{k\geqslant 2} (k\beta_k^0)^2 \leqslant (\mu\rho)^{1/2} |\beta_1^0|^{3/2}, \quad 0 < |\beta_1^0| \leqslant \mu\rho.$$

Moreover,

$$|\dot\beta_1(0) - \dot{\overline{\beta}}_1(0)| \leqslant \mu, \quad \sqrt{\sum_{k\geqslant 2} |\dot\beta_k(0) - \dot{\overline{\beta}}_k(0)|^2} \leqslant \mu,$$

where

$$\overline{\beta}_1{}^2(t) = \frac{1}{t+t_0} \left(t_0 \beta_1^2(0) + 2 \sum_{k\geqslant 2} \beta_k^2(0) \int_0^t \frac{d\tau}{(1+\tau/t_0)^{k+2}} \right), \tag{1.14}$$

$$\overline{\beta}_k(t) = \frac{\beta_k(0)}{(1+t/t_0)^{k/2+1}}. \tag{1.15}$$

Furthermore, there exists a constant C (*slightly exceeding* 1) *such that for any* $t \geqslant 0$

$$|\dot\beta_1(t) - \dot{\overline{\beta}}_1(t)| \leqslant C\mu, \quad \sqrt{\sum_{k\geqslant 2} |\dot\beta_k(t) - \dot{\overline{\beta}}_k(t)|^2} \leqslant C\mu.$$

Remark 1.1. Formula (1.15) clarifies the nature of the above-mentioned result. If for $qt < 0$, i.e., in the case of sink, the problem (1.1)–(1.3) is solvable for any small t (in our case $q = 2 > 0$, this means for any small negative t), then the initial contour Γ_0 is necessarily analytic. Indeed, by (1.15), the function (1.12) is defined for any small $t < 0$ only if the Fourier coefficients $\beta_k^0 = \beta_k(0)$ of the function (1.13) exponentially rapidly decrease and thereby define (cf., for example, [28, Section 12]) an analytic function.

We also note that the last assertion (concerning a time-infinite "drift" of the perturbed circle; cf. [29]) agrees with the following fact: if, under a deformation $t \mapsto \Gamma_t$, for some finite $t_* \neq 0$ the curve Γ_{t*} is homothetic to a circle centered at the origin $\{0\}$, i.e., at the point where soruce/sink is located, then the curve Γ_0, and, consequently, all the curves Γ_t are homothetic to the same circle.

2 Quasicontour Model. Critical Manifold. Attractor–Repeller

Theorem 1.2 clarifies the evolution of weak perturbations of the circle (represented partially in Figure 1). In the case of an arbitrary initial contour, Equation (1.11) was treated within the framework of the so-called quasicontour model [30]– [32] (also referred to as a polygonal model [34] or a finite point model [33]) of the problem (1.1)–(1.3).

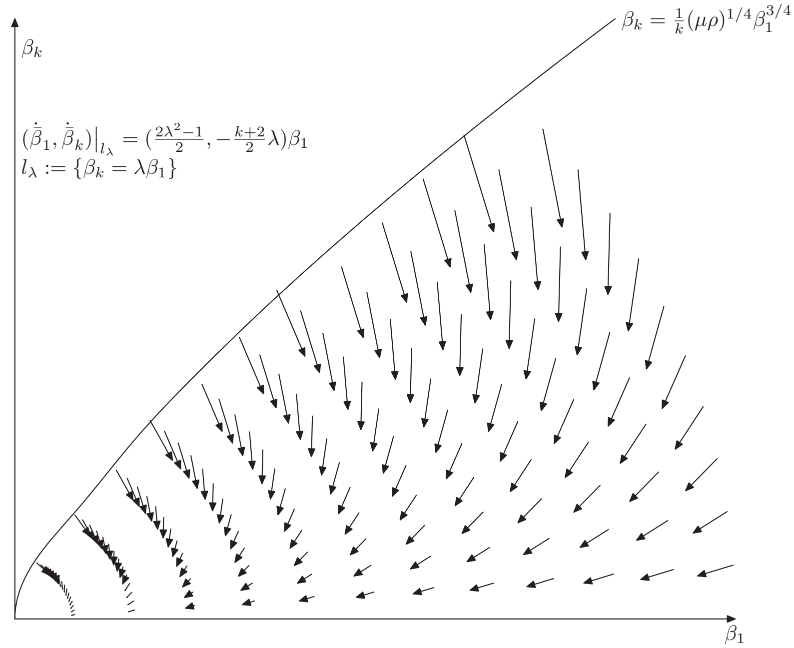

FIG. 1. The vector field $(\dot{\bar{\beta}}_1, \dot{\bar{\beta}}_k)$ in the case $\beta_j(0) = 0$ for $j \neq 1$ and $j \neq k$.

2.1. A quasicontour model of the problem (1.1)–(1.3) differs from the problem itself only by the following properties.

First, the class of smooth curves Γ_t is replaced with the class of quasicontours. Such quasicontours are, in particular, presented by polygonal contours Γ_t^m bounding polygonal domains Ω_t^m that are symmetric (as well as the domain Ω_t) with respect to the Ox-axis. The ordinates of vertices $s_0, s_{\pm 1}, \ldots, s_{\pm m}$ of Γ_t^m have the sign of their indices, in particular, the vertex s_0 lies on the positive half Ox-axis. These vertices form $2m+1$ sides $[s_0, s_{\pm 1}]$, $[s_{\pm 1}, s_{\pm 2}]$, $\ldots$, $[s_{\pm(m-1)}, \ s_{\pm m}]$, $[s_m, s_{-m}]$.

Second, the evolution of these sides is determined by the motion of vertices of the quasicontour Γ_t^m in accordance with the conditions (2.1) (inheriting the Stokes condition (1.2) and the dynamic Leibenson condition (1.3)) and (2.3) (inheriting the kinematic condition (1.1)).

We introduce a vector-valued function $\mathbf{N} : t \mapsto \mathbf{N}(t) = (N_1(t), \ldots, N_m(t))$ and a vector $\sigma = (\sigma_1, \ldots, \sigma_m)$ in terms of which the quasicontour Γ_t^m is given. Hereinafter, we denote by $N_j(t)$ the angle of slope to the Ox-axis of the outward normal to the jth side $[s_{j-1}, s_j]$ of the quasicontour Γ_t^m. Moreover, [1] $-\pi/2 < N_1 < \pi/2$, $-\pi < N_k - N_{k-1} < \pi$ for all $k \in \{2, \ldots, m\}$, $0 < N_m < 2\pi$ for $j = 1, \ldots, m$. Further, the parameter $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_m)$ is uniquely defined by

[1] The angle of slope of the outward normal to the remaining sides of the quasicontour is defined by symmetry respect to the Ox-axis. In particular, the angle of slope to the $(m+1)$th side $[s_m, s_{-m}]$ is equal to π.

the initial contour Γ_0^m via the solution to the problem

$$u_{xx} + u_{yy} = 2\delta(x, y) \quad \text{in} \quad \Omega_t^m, \qquad u = 0 \quad \text{on} \quad \Gamma_t^m \tag{2.1}$$

for $t = 0$ and determines, by means of formula (2.2), the length of sides of the quasicontour Γ_t^m. Namely, σ_k is the value of the harmonic conjugate v of u in $\Omega_t^m \cap \mathbb{R}_+^2$ for which the kth vertex $s_k \in \Gamma_t^m$ belongs to the level curve $\{(x, y) \in \Omega_t^m \mid v(x, y) = \sigma_k\}$. It is obvious that $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_m < \sigma_{m+1} = 1$ and the coordinates (x_k, y_k) of the vertex s_k (for $k = 1, \ldots, m$) are found in accordance with (1.6) by the formula

$$x_k = x_0 - \int_0^{\sigma_k} e^a \sin b \, dv, \qquad y_k = \int_0^{\sigma_k} e^a \cos b \, dv, \tag{2.2}$$

where

$$x_0 = \int_{-\infty}^{0} e^{A(t;u,0)} \, du, \qquad a(t, v) + ib(t, v) = A(t; 0, v) + iB(t, 0, v),$$

and the Helmholtz–Kirchhoff function

$$A + iB : w = u + iv \mapsto A(t; u, v) + iB(t; u, v) = \ln \left| \frac{\partial z(t; w)}{\partial w} \right| + i \arg \frac{\partial z(t; w)}{\partial w}$$

corresponds for $z = x + iy \in \Omega_t^m \cap \mathbb{C}_+$ (cf. (1.5)) to the solution $u : (x, y) \mapsto u(x, y)$ to the problem (2.1).

For the kinematic condition on the quasicontour Γ_t^m, taking into account (1.1)$\Leftrightarrow$(1.8), we obtain it by

$$R_k = \dot{x}_k \cos b_k + \dot{y}_k \sin b_k, \qquad k = 0, \ldots, m, \tag{2.3}$$

where $b_k = b(t, \sigma_k)$ and R_k is the “normal” velocity of the points $s_k \in \Gamma_t^m$; more exactly, the algebraic value of the projection of the velocity of $s_k \in \Gamma_t^m$ onto the outward “normal” to Γ_t^m which is the bisector of the exterior angle at the vertex s_k of the polygon Γ_t^m. Setting $N_0 \stackrel{\text{def}}{=} -N_1$, we note that $b_0 = 0$, $b_k = (N_k + N_{k+1})/2$, $k = 0, 1, \ldots, m$.

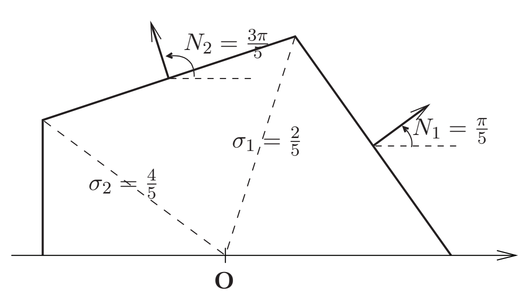

FIG. 2. The upper part of the regular pentagonal quasicontour Γ_t^5 with source/sink at the center.

The following assertion is obvious.

Proposition 2.1. *A quasicontour Γ_0 is a regular $(2m+1)$-polygon with a source/sink located at its center if and only if $(\mathbf{N}, \boldsymbol{\sigma}) = (\widehat{\mathbf{N}}, \widehat{\boldsymbol{\sigma}}) \stackrel{def}{=} (\widehat{N}_1, \ldots, \widehat{N}_m; \widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$, where*

$$
\begin{aligned}
&2\widehat{N}_1 = \widehat{N}_2 - \widehat{N}_1 = \cdots = \widehat{N}_{m+1} - \widehat{N}_m = (2\pi)/(2m+1),\\
&\widehat{\sigma}_1 = \widehat{\sigma}_2 - \widehat{\sigma}_1 = \cdots = \widehat{\sigma}_m - \widehat{\sigma}_{m-1} = 2/(2m+1).
\end{aligned}
$$

By the aforesaid, the quasicontour Γ_t^m is uniquely determined by the parameter $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\Gamma_0^m)$ and the vector-valued function $\mathbf{N} : t \mapsto \mathbf{N}(t)$.

Theorem 2.1 (cf. [32])**.** *The vector-valued function $\mathbf{N} : t \mapsto \mathbf{N}(t) = (N_1(t), \dots, N_m(t))$ is a solution to the matrix equation*

$$
Q(\mathbf{N}, \boldsymbol{\sigma})\dot{\mathbf{N}} = \exp(-2\alpha_0)\, P^0(\mathbf{N}, \boldsymbol{\sigma}) + \dot{\alpha}_0\, P^1(\mathbf{N}, \boldsymbol{\sigma}),
$$

where

$$
\begin{aligned}
e^{-2\alpha_0(t)} &= (t + |\Omega_0^+|) \int_0^1 \left(\int_{-\infty}^0 e^{2(\pi u + \sum\limits_{k\geqslant 1} \beta_k(t) e^{\pi k u} \cos \pi k v)}\, du \right) dv,\\
\beta_k(t) &= \frac{2}{k}\Big(N_1(t) + \sum_{j=1}^m \frac{N_{j+1}(t) - N_j(t)}{\pi} \cos k\pi\sigma_j \Big),
\end{aligned}
$$

and the entries of $Q = (q_{kj})_{1\leqslant k,j\leqslant m}$, $P^0 = (p_k^0)_{1\leqslant k\leqslant m}$, and $P^1 = (p_k^1)_{1\leqslant k\leqslant m}$ take the form

$$
q_{kj} = \frac{1}{\pi}\left[\tan\frac{N_k - N_{k-1}}{2} \int_0^{\sigma_{k-1}} \frac{\ln f_j(\eta)}{g(\mathbf{N},\eta)}\, d\eta + \tan\frac{N_{k+1} - N_k}{2} \int_0^{\sigma_k} \frac{\ln f_j(\eta)}{g(\mathbf{N},\eta)}\, d\eta \right] + \delta_{kj} \int_{\sigma_{k-1}}^{\sigma_k} \frac{d\eta}{g(\mathbf{N},\eta)},
$$

$$
\begin{aligned}
p_k^0(\mathbf{N}) =\,& d_{k-1}(\mathbf{N}) \cos\frac{N_k - N_{k-1}}{2} - d_k(\mathbf{N}) \cos\frac{N_{k+1} - N_k}{2}\\
&+ (\delta_{k1} - 1)\, \tan\frac{N_k - N_{k-1}}{2} \sum_{j=1}^{k-1} \left[d_{j-1}(\mathbf{N}) \sin\frac{N_j - N_{j-1}}{2} + d_j(\mathbf{N}) \sin\frac{N_{j+1} - N_j}{2} \right]\\
&- \tan\frac{N_{k+1} - N_k}{2} \sum_{j=1}^{k} \left[d_{j-1}(\mathbf{N}) \sin\frac{N_j - N_{j-1}}{2} + d_j(\mathbf{N}) \sin\frac{N_{j+1} - N_j}{2} \right],
\end{aligned}
$$

$$
p_k^1(\mathbf{N}) = \tan\frac{N_k - N_{k-1}}{2} \int_0^{\sigma_{k-1}} \frac{d\eta}{g(\mathbf{N},\eta)} + \tan\frac{N_{k+1} - N_k}{2} \int_0^{\sigma_k} \frac{d\eta}{g(\mathbf{N},\eta)},
$$

where δ_{kj} is the Kronecker symbol,

$$
\begin{aligned}
f_j(\eta) &= \left| \frac{\sin\frac{\pi}{2}(\eta + \sigma_{j-1})\, \sin\frac{\pi}{2}(\eta - \sigma_{j-1})}{\sin\frac{\pi}{2}(\eta + \sigma_j)\, \sin\frac{\pi}{2}(\eta - \sigma_j)} \right|, \quad 1 \leqslant j \leqslant m,\\
f_{m+1}(\eta) &= 2\left| \sin\frac{\pi}{2}(\eta + \sigma_m)\, \sin\frac{\pi}{2}(\eta - \sigma_m) \right|,\\
g(\mathbf{N},\eta) &= 2\prod_{k=1}^{m+1} f_k(\eta)^{N_k/\pi} \qquad d_k(\mathbf{N}) = \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} g(\mathbf{N},\eta)\, d\eta,\\
\lambda_k &= \frac{\sigma_{k-1} + \sigma_k}{2}, \quad 1 \leqslant k \leqslant m,\ \lambda_0 = -\lambda_1,\ \lambda_{m+1} = 1.
\end{aligned}
$$

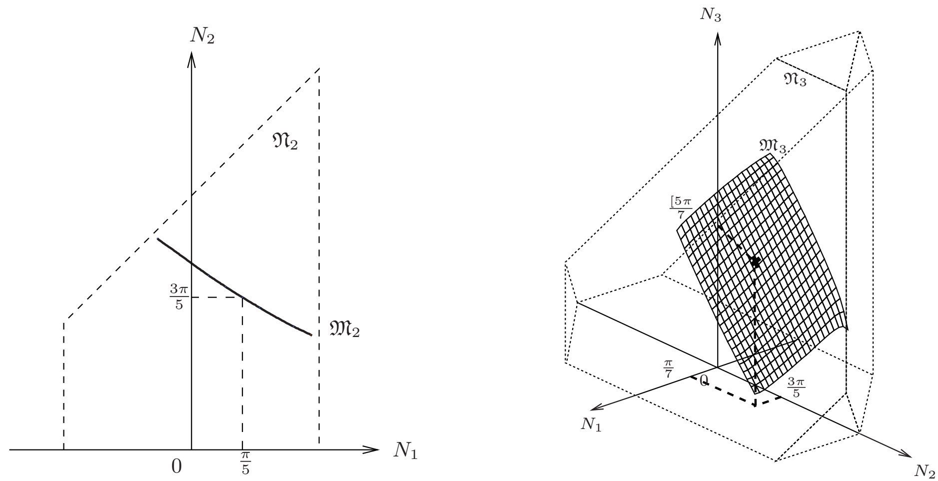

FIG. 3. The critical manifold $\mathfrak{M}_2(\mathbf{N}, \widehat{\boldsymbol{\sigma}})$ and $\mathfrak{M}_3(\mathbf{N}, \widehat{\boldsymbol{\sigma}})$ in the spaces $\mathfrak{N}_2(\mathbf{N}, \widehat{\boldsymbol{\sigma}})$ and $\mathfrak{N}_3(\mathbf{N}, \widehat{\boldsymbol{\sigma}})$ respectively.

2.2. The numerical analysis in [30]–[32] shows that, in the space $\mathfrak{N}_m(\mathbf{N}, \boldsymbol{\sigma})$ of quasicontours Γ_t^m, there is a manifold of codimension 1 $\mathfrak{M}_m(\mathbf{N}, \boldsymbol{\sigma}) = \{(\mathbf{N}, \boldsymbol{\sigma}) \in \mathfrak{N}_m(\mathbf{N}, \boldsymbol{\sigma}) \big| \det Q(\mathbf{N}, \boldsymbol{\sigma}) = 0\}$, called *critical* in [32] because it is connected with the degeneracy of the matrix $Q(\mathbf{N}, \boldsymbol{\sigma})$ corresponding to the operator $\mathbf{I} - \mathbf{K}(\beta)$ in (1.11).

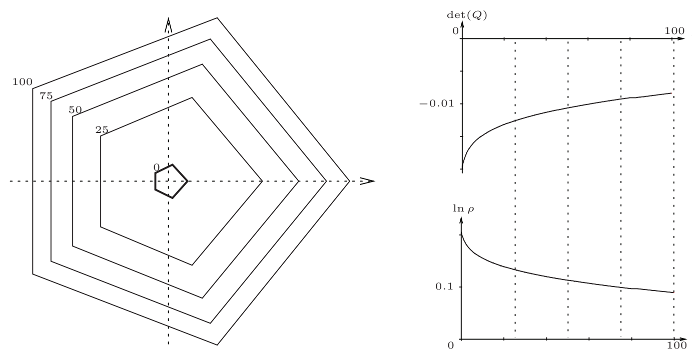

FIG. 4. The evolution of a weakly perturbed quasicontour to the regular quasicontour Γ_t^5 with a centered source in the case $0 < N_1^0 - \pi/5 \ll 1, 0 < N_2^0 - 3\pi/5 \ll 1, \det Q(\mathbf{N}^0) < 0$.

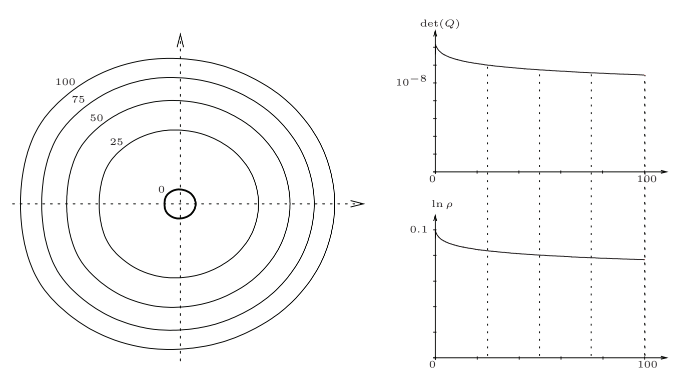

FIG. 5. The evolution of a weakly perturbed quasicontour to the (spline-interpolation) regular quasicontour Γ_t^7 with a centered source.

Figures 4 and 5 present the numerical results, including the graphs $t \mapsto \det Q(t)$ and $t \mapsto \ln \rho(t)$, where ρ is the maximal possible ratio between the distances from two points on the quasicontour to the source/ sink. These results show that the regular quasicontours $\Gamma_t^5 \in \mathfrak{M}_5(\mathbf{N}, \boldsymbol{\sigma})$ and $\Gamma_t^7 \in \mathfrak{M}_7(\mathbf{N}, \boldsymbol{\sigma})$ at the center of which there is a source/sink are attractors in the case of source (and repellers in the case of sink). Theorem 1.2 illustrated by Figure 1 leads to the same conclusion relative to the circle (which is regarded as a regular quasicontour with infinitely many sides) with a source/sink at its center.

Regarding an evolution of small perturbations of other points of the critical manifold, the points $\mathfrak{M}_m(\mathbf{N}, \sigma)$, where calculations were performed, turned out to be attractors in the case of sink (thereby repellers in the case of source). In particular, the results presented in Figures 6 and 7 an information about the evolution of a decentered "circle" (i.e., the regular decentered quasicontour Γ_t^8) in the case of sink. We note that the results presented in these figures completely agree with the above-mentioned result of [6].

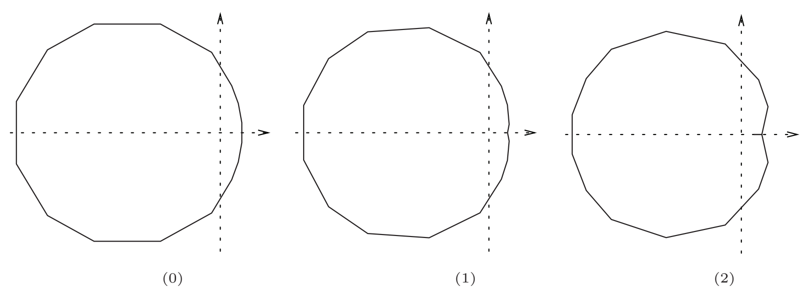

Fig. 6. Three stages of evolution in the case of sink of the decentered Kufarev "circle" (the decentered quasicontour Γ_t^8).

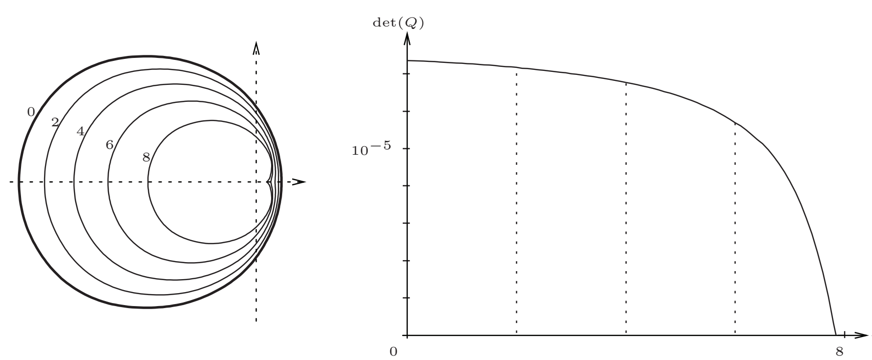

Fig. 7. The evolution in the case of sink of the decentered Kufarev "circle" (the decentered spline-interpolation quasicontour Γ_t^8.

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